Language Inclusion Algorithms as Complete Abstract Interpretations

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Abstract Interpretation from Büchi Automata

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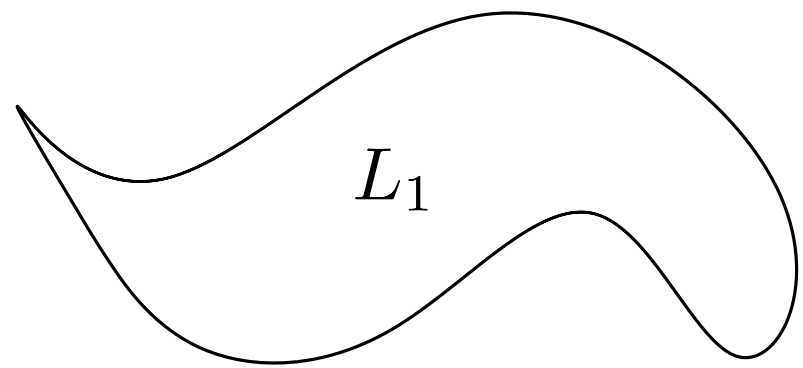
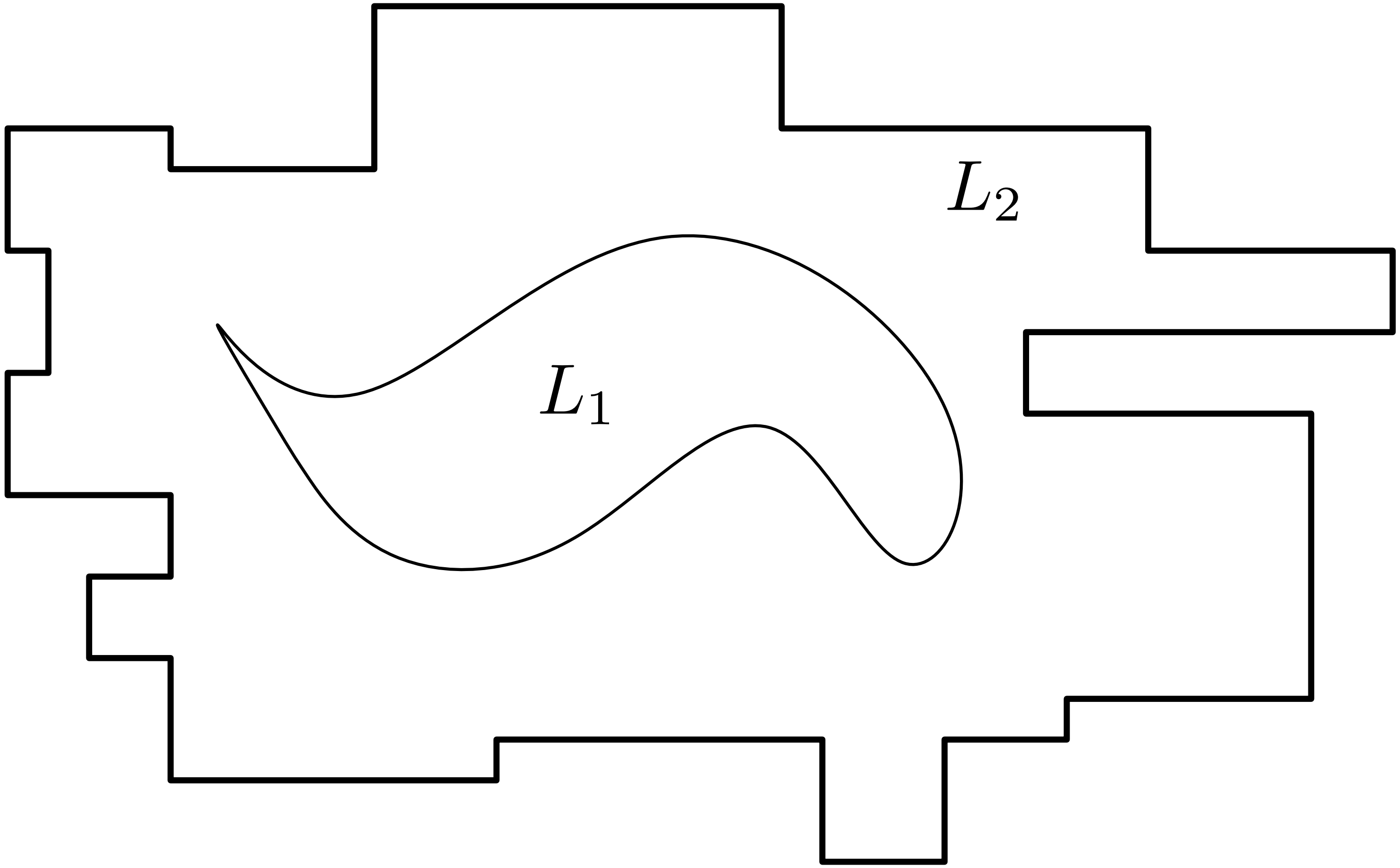
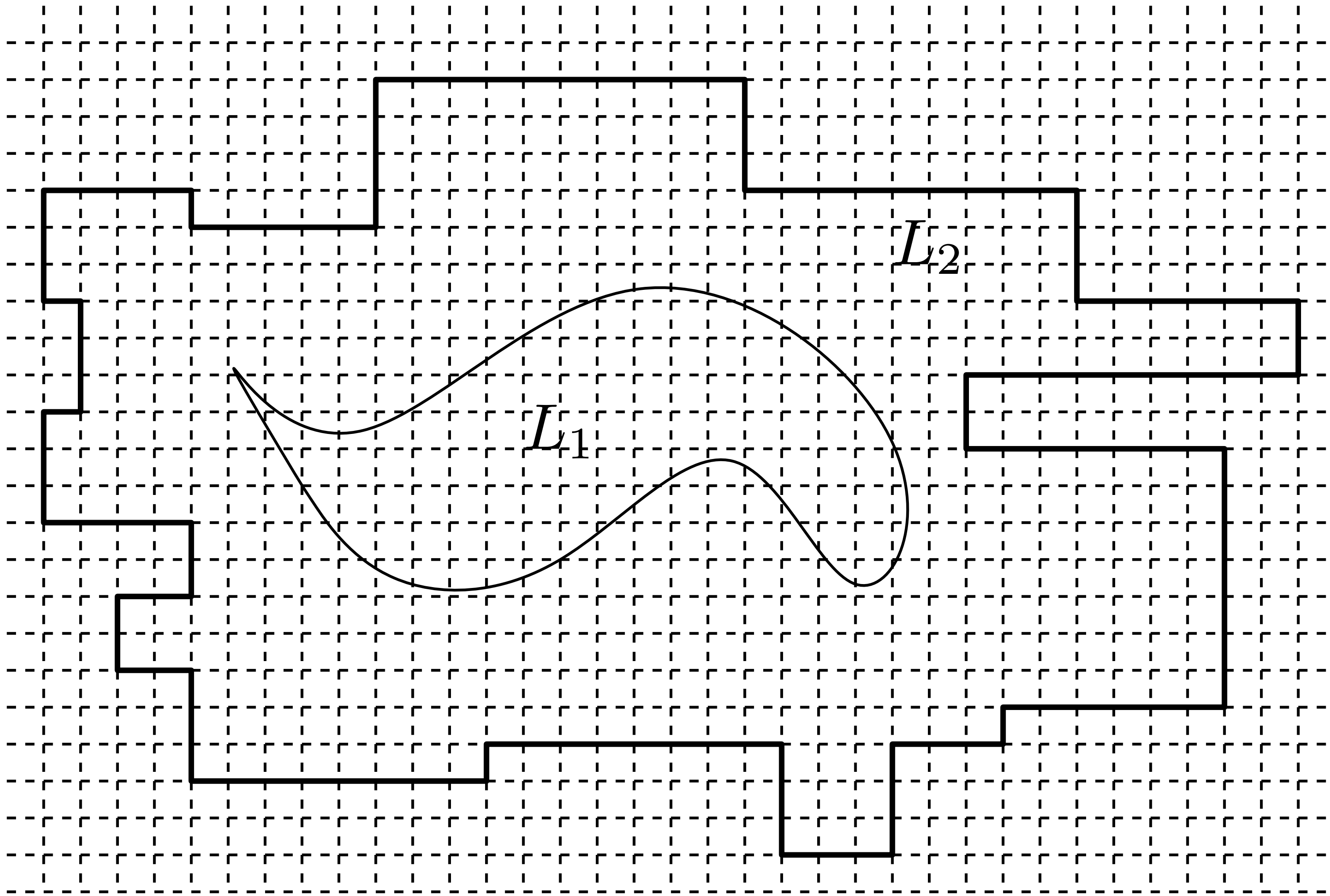

L_1



L_2
L_1



L_2
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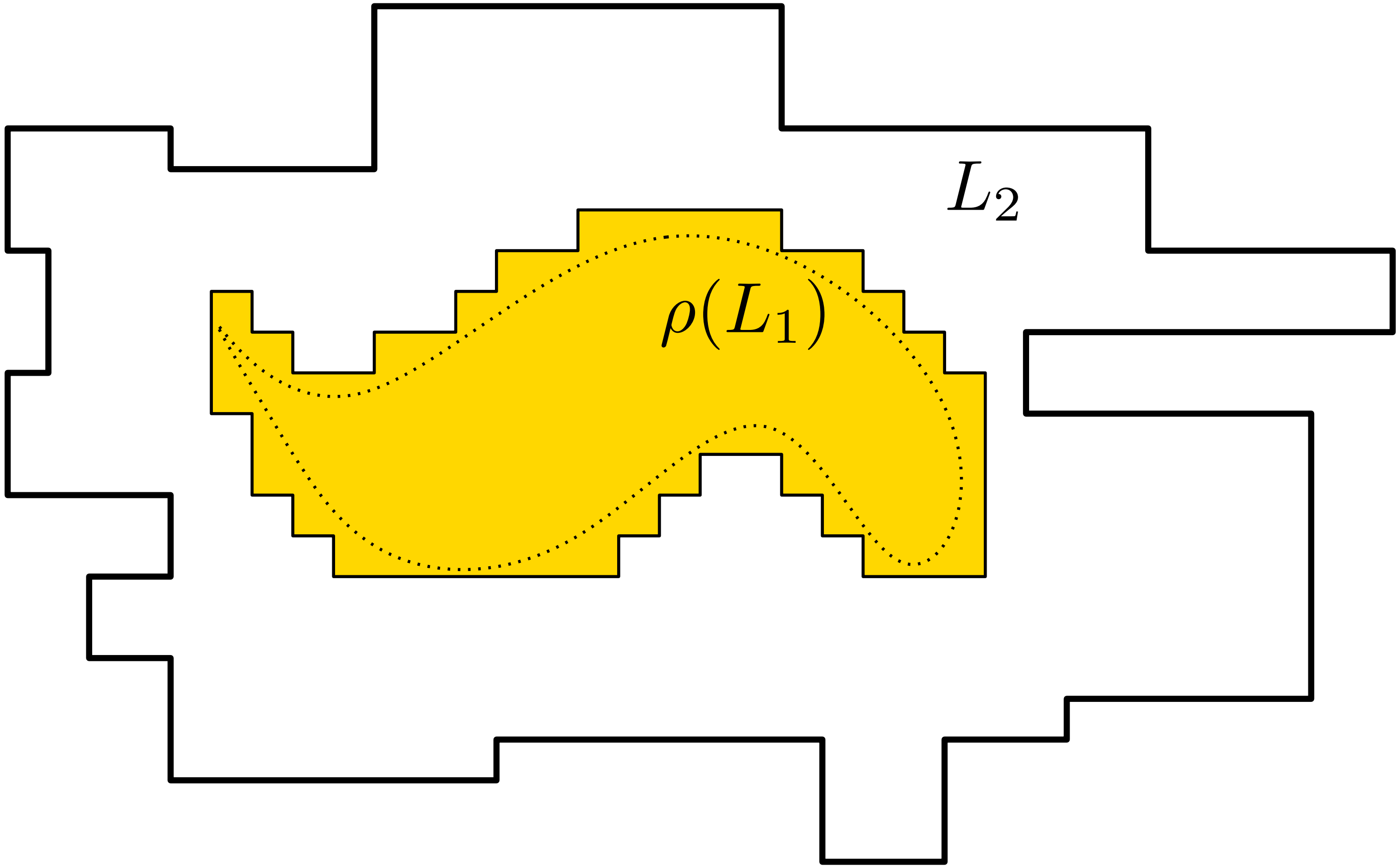
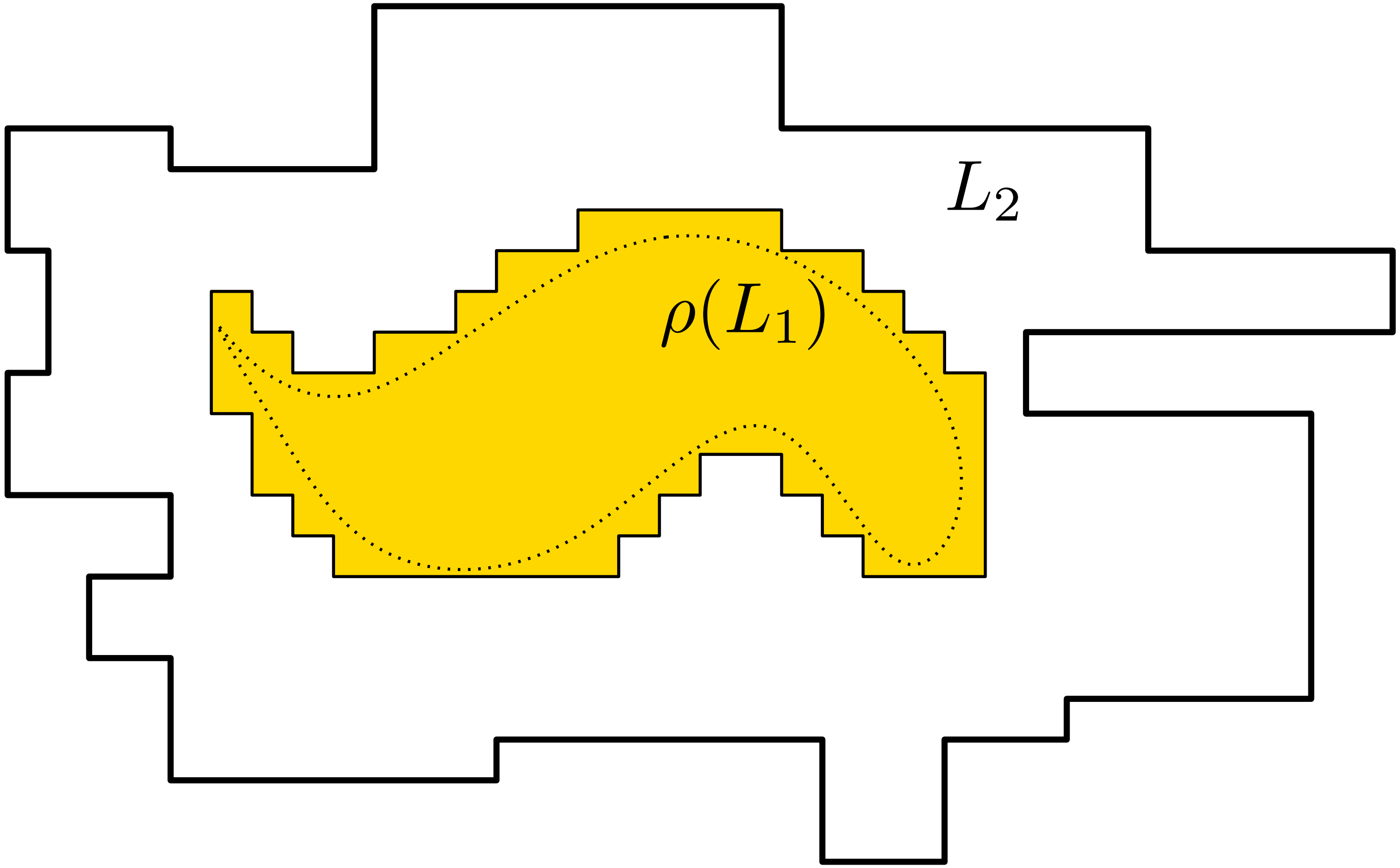
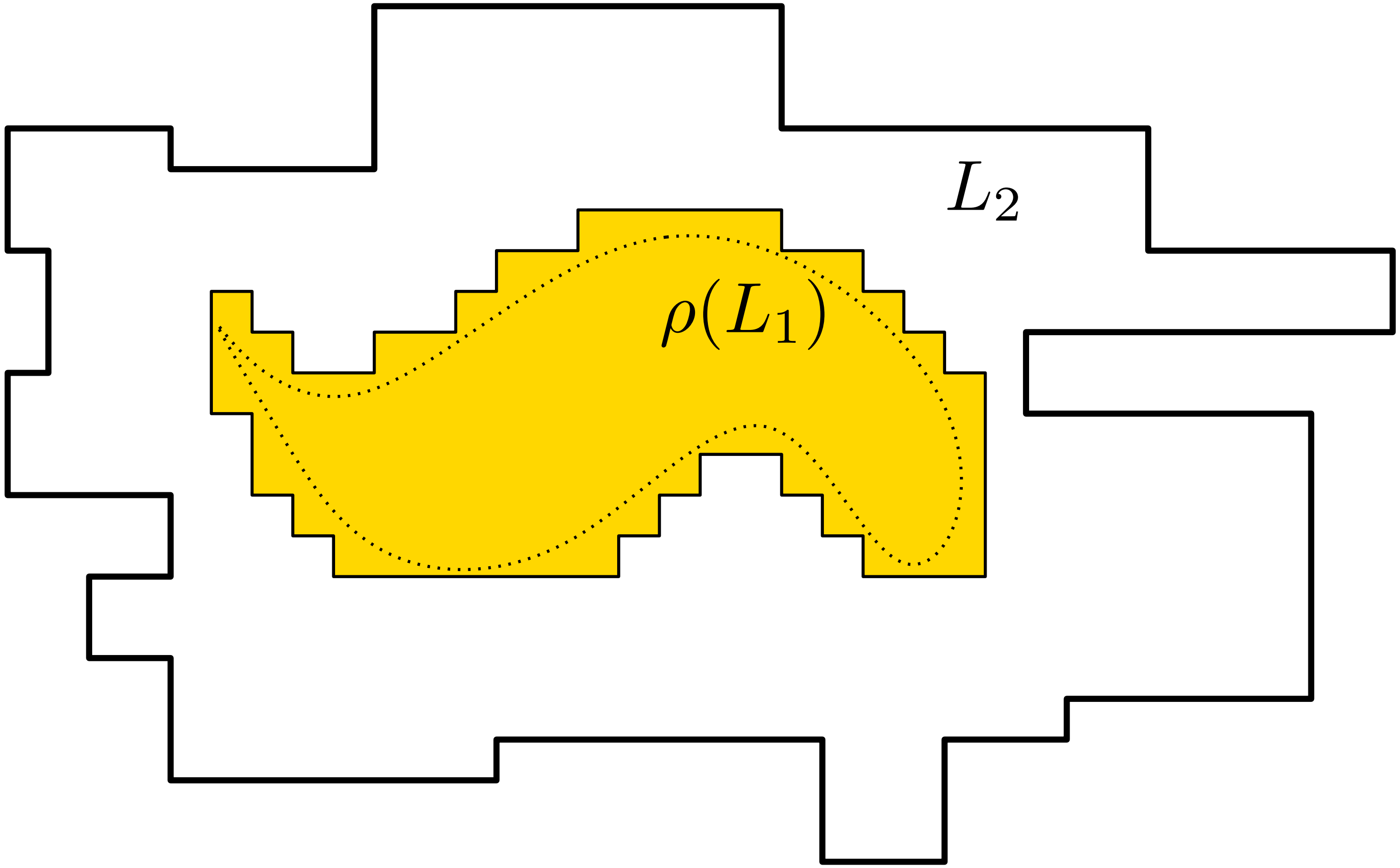

L_2
$\rho(L_1)$

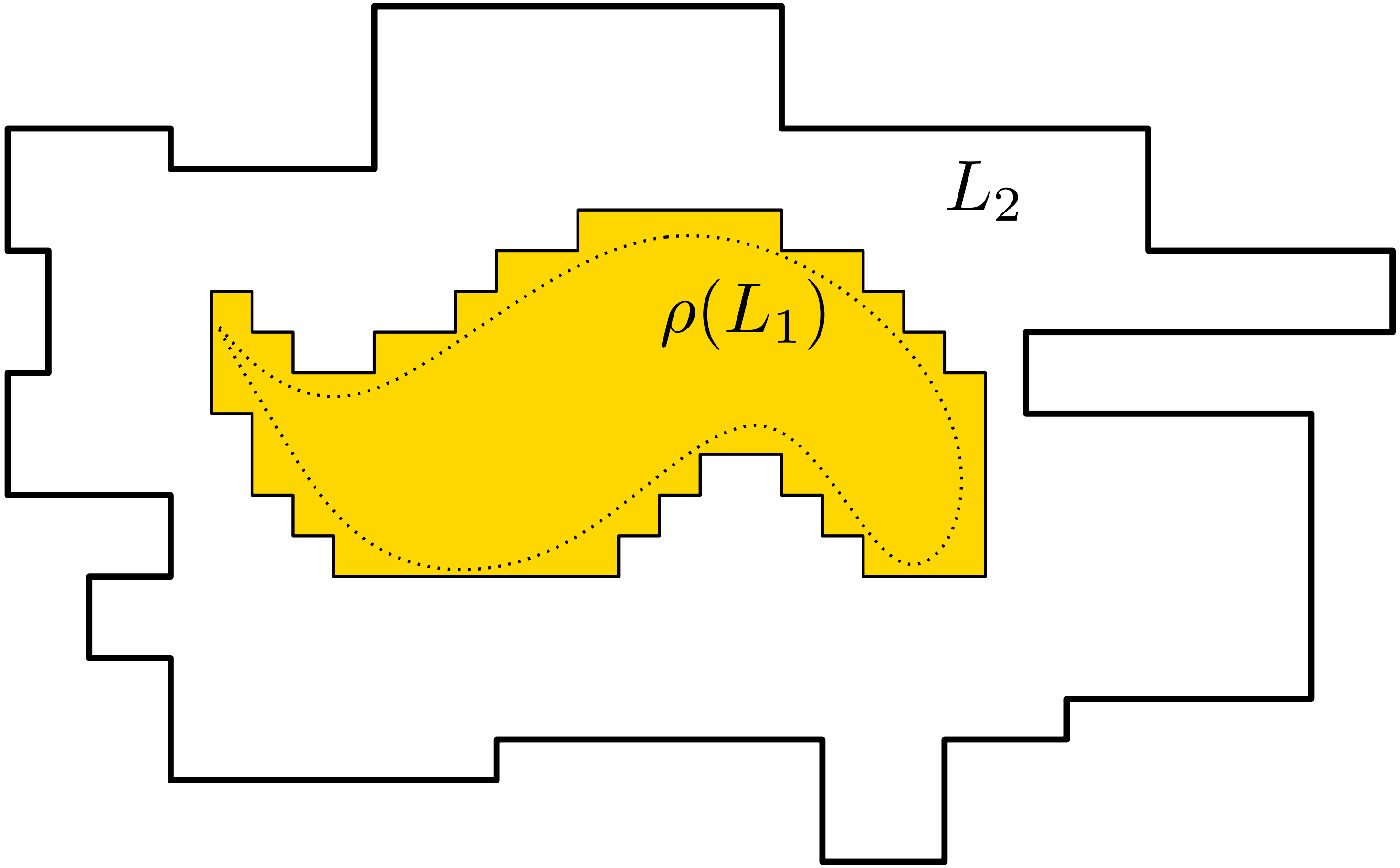


- ρ is extensive: $L \subseteq \rho(L)$
- ρ is monotone: $L \subseteq L'$ implies $\rho(L) \subseteq \rho(L')$
- ρ is idempotent: $\rho(\rho(L)) = \rho(L)$



Assume $\rho(L_2) = L_2$, then

$$L_1 \subseteq L_2 \text{ iff } \rho(L_1) \subseteq L_2$$



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L_1 as a fixed point

$$X \to aX \mid \epsilon$$

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$\varnothing$

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$\varnothing$ $\{\epsilon\}$

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$\varnothing$ $\{\epsilon\}$ $\{\epsilon, a\}$

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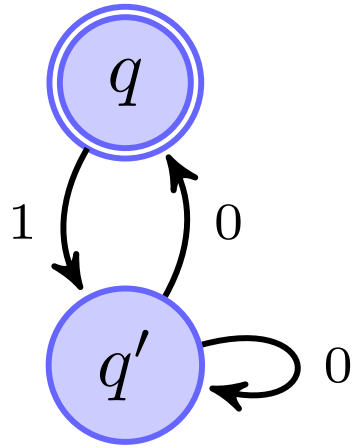
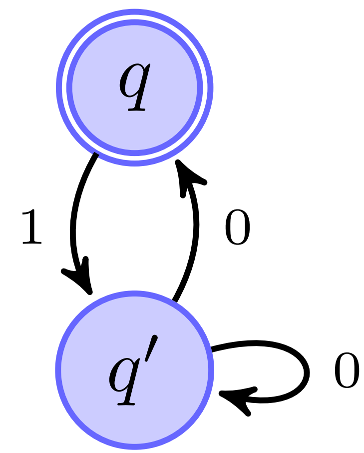
L_1 as a fixed point (cont'd)



$$\mathrm{lfp}\left(\lambda\begin{pmatrix} X_q \\ X_{q'} \end{pmatrix}.\begin{pmatrix} \{\epsilon\} & \cup & 1X_{q'} \\ \varnothing & \cup & 0X_q \cup 0X_{q'} \end{pmatrix}\right)$$

L_1 as a fixed point (cont'd)



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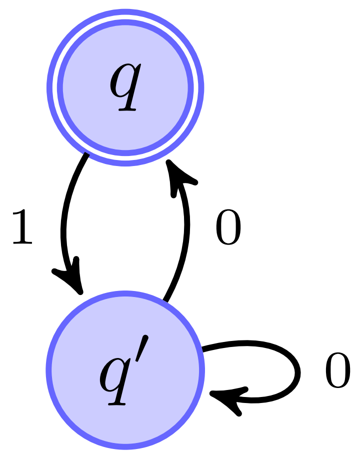

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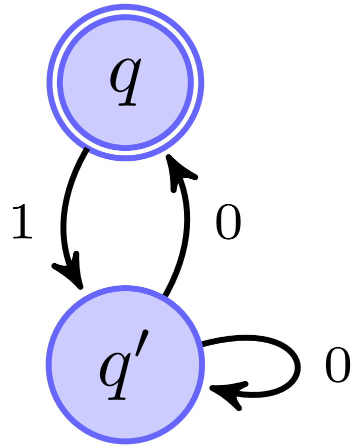

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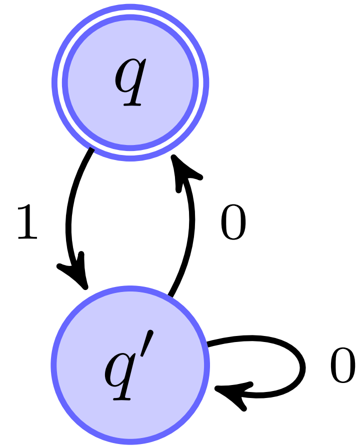

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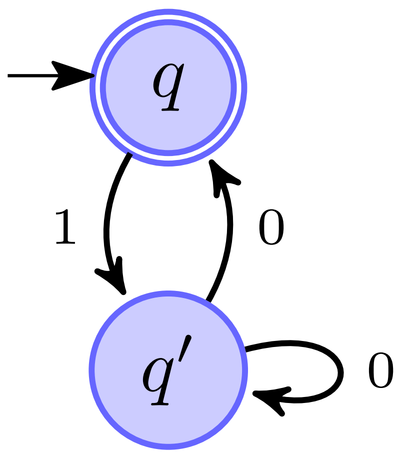

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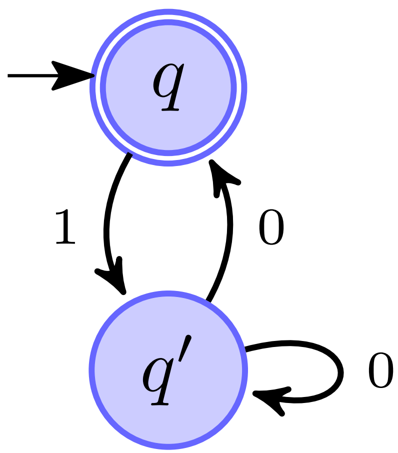
$$L_1 = L_q = (10^+)^* = \langle \mathrm{lfp}\,\lambda X.\, b \cup Fn(X) \rangle_q$$

L_1 as a fixed point (cont'd)



b $\quad$ Fn

$$\mathrm{lfp}\left(\lambda \begin{pmatrix} X_q \\ X_{q'} \end{pmatrix} . \begin{pmatrix} \{\epsilon\} & \cup & 1X_{q'} \\ \emptyset & \cup & 0X_q \cup 0X_{q'} \end{pmatrix}\right)$$

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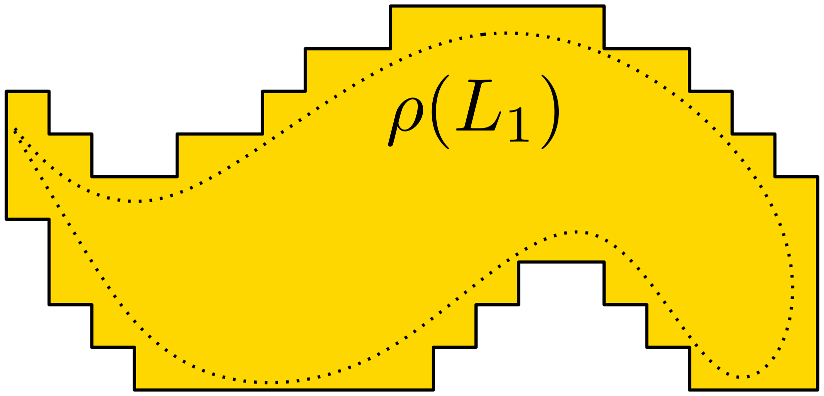

$$\rho(L_1) = \rho\big(\mathrm{lfp}\,\lambda X.\, b \cup Fn(X)\big)$$

if ρ is complete for Fn (i.e. $\rho\ Fn\ \rho = \rho\ Fn$) then

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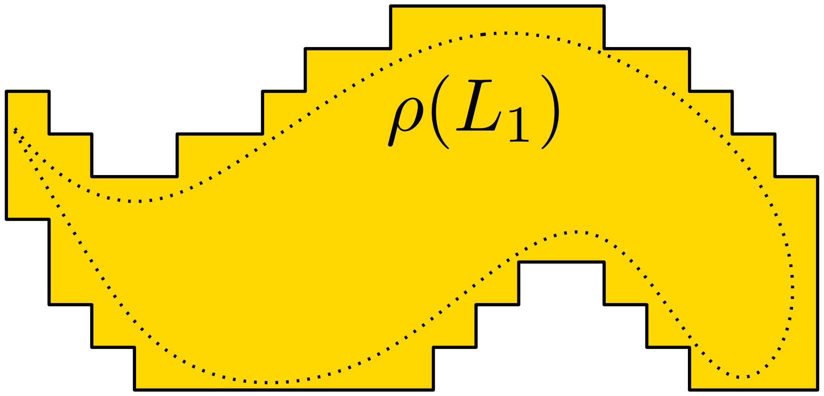

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Wanted

$\rho(L_2) = L_2$

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Quasiorder induced ρ

Given a quasiorder $\leqslant$ on words, define

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Example of quasiorder

Nerode left quasiorder relative to L_2 [Varricchio,deLuca'94]

$$x \leqslant^{l}_{L_2} y \Leftrightarrow L_2\, x^{-1} \subseteq L_2\, y^{-1}$$

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$\leqslant^{l}_{L_2}$ induce the coarsest qo based abstraction $\rho_{\leqslant^{l}_{L_2}}$

Deciding $L_1 \subseteq L_2$

$$L_1 \subseteq L_2 \quad \text{iff} \quad \rho_{\leqslant^l_{L_2}}(L_1) \subseteq L_2$$

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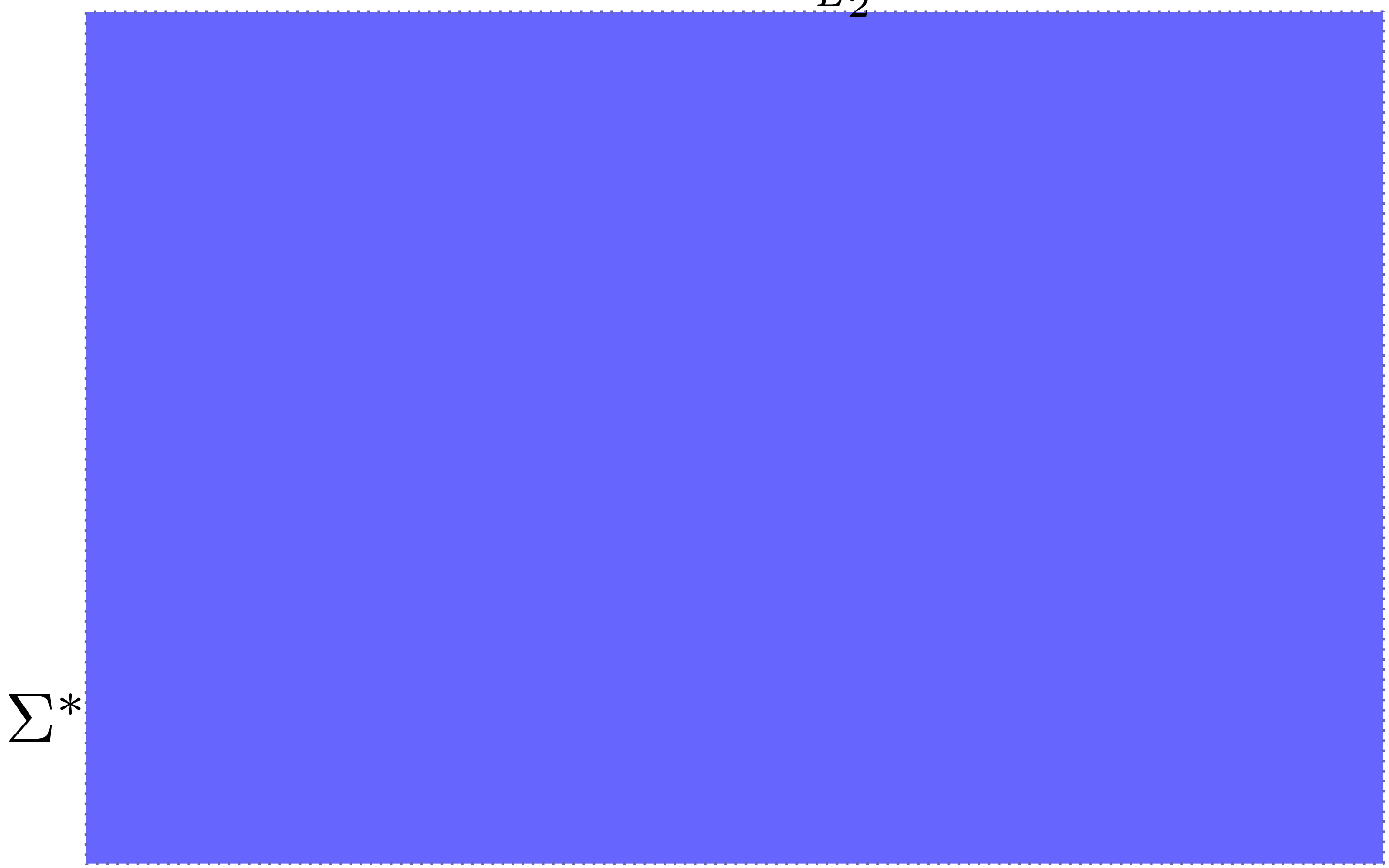

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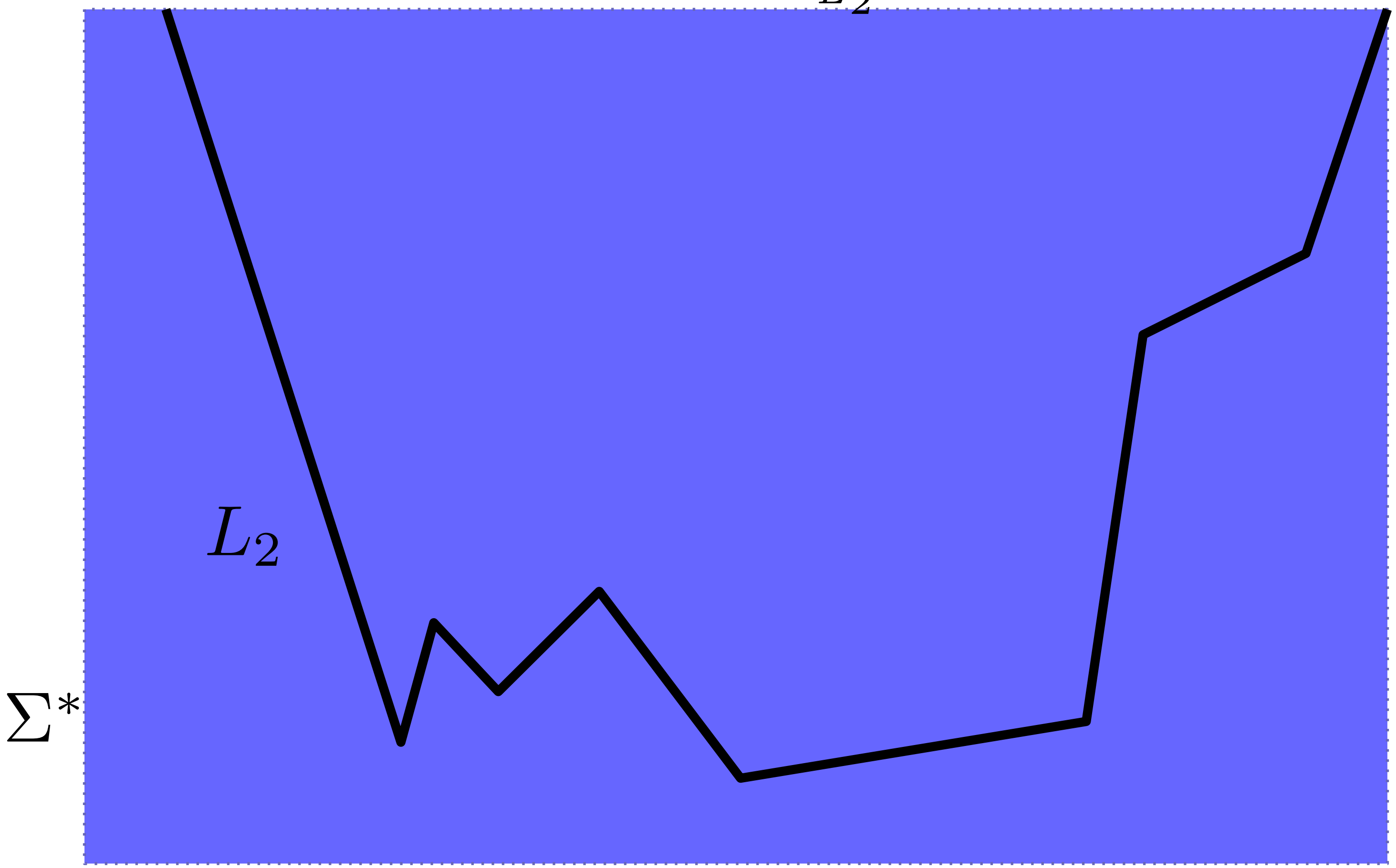

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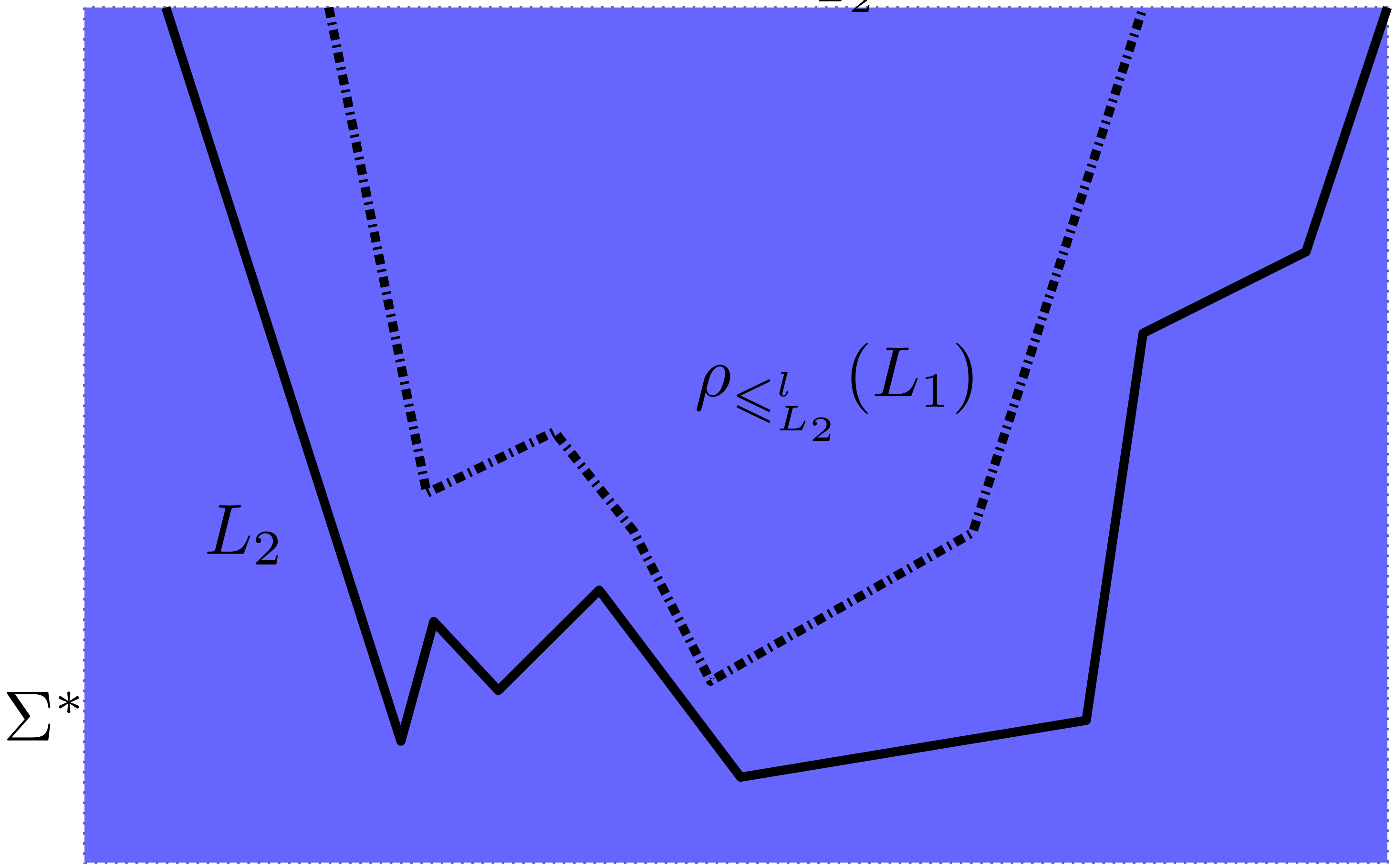

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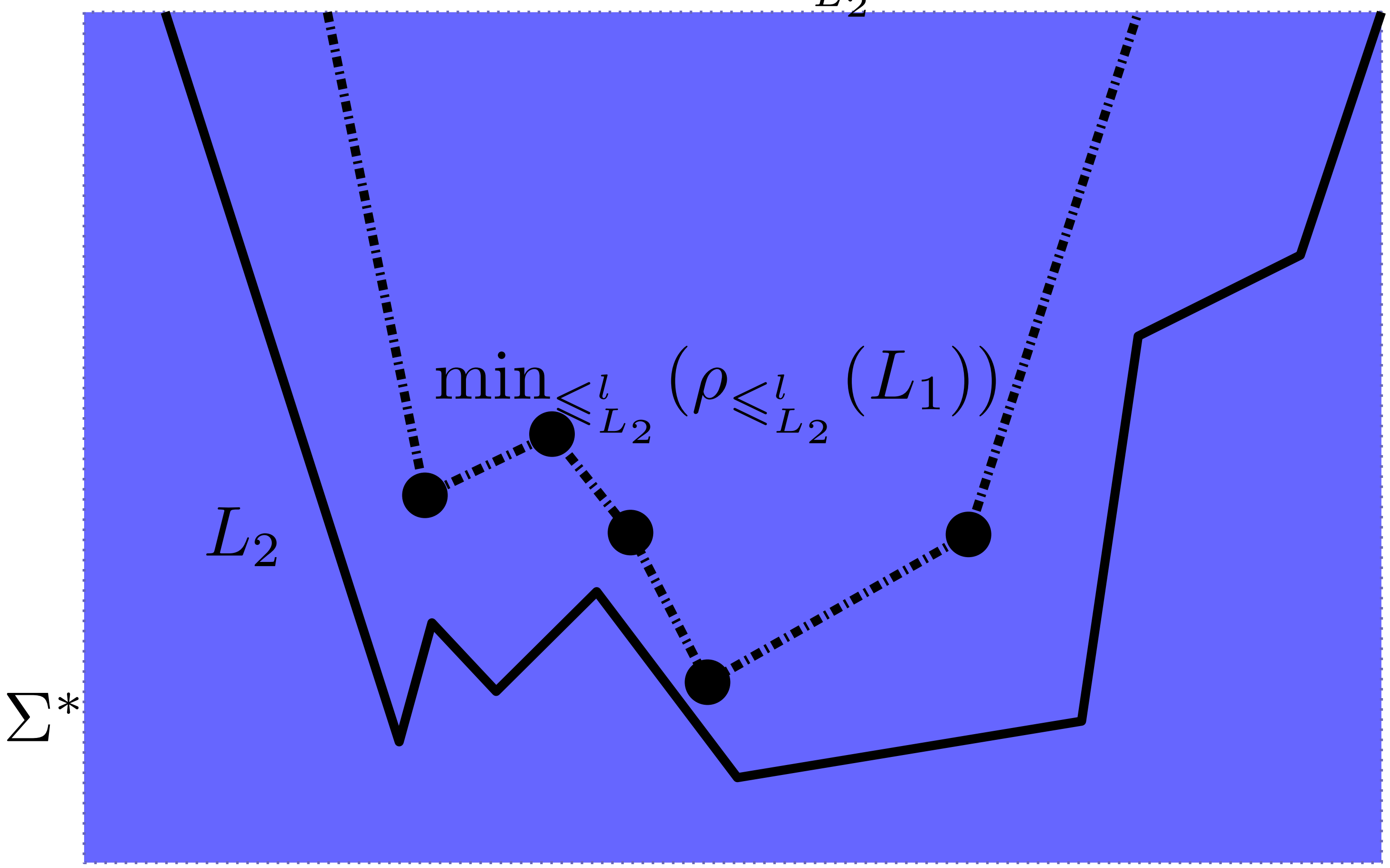

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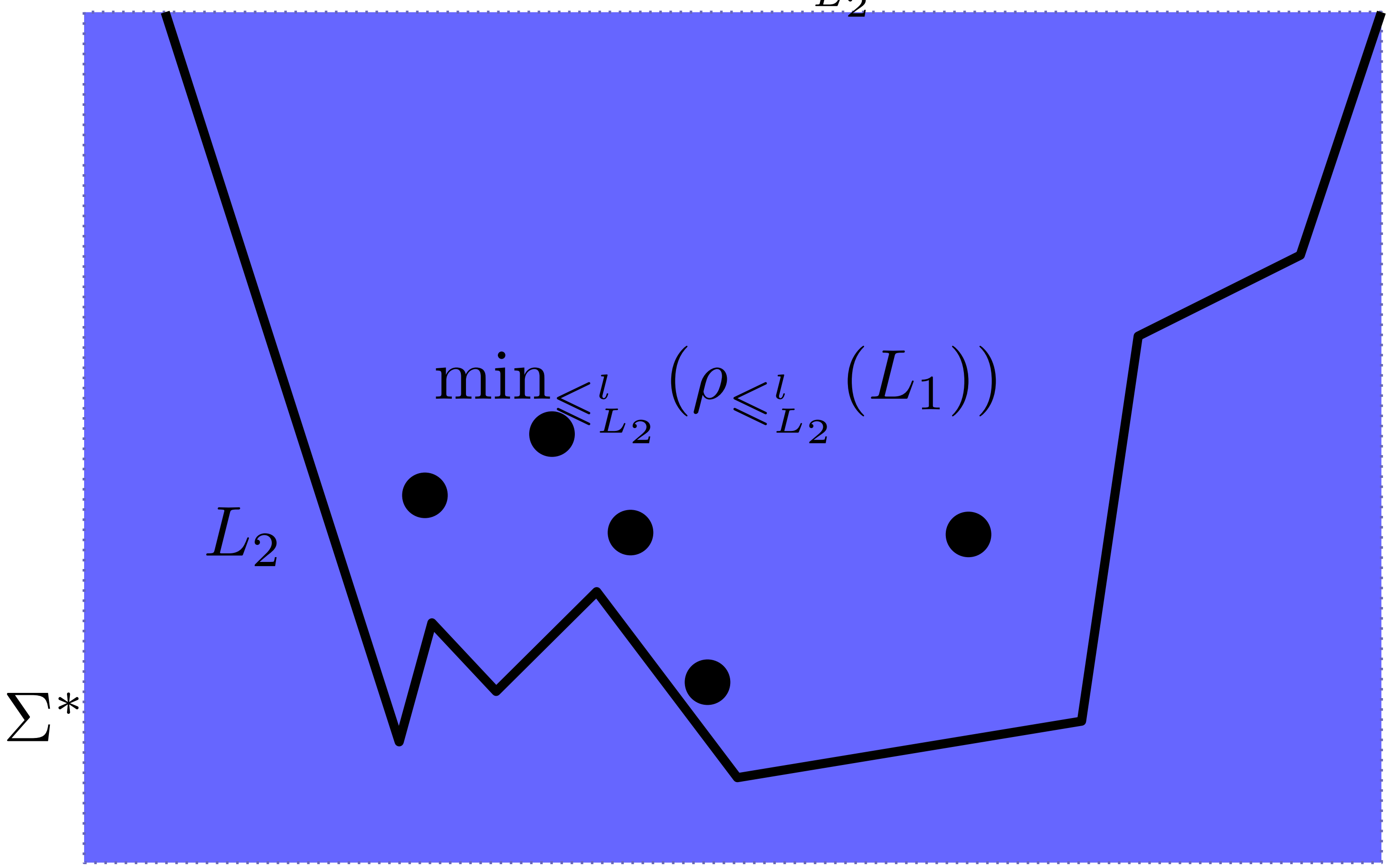

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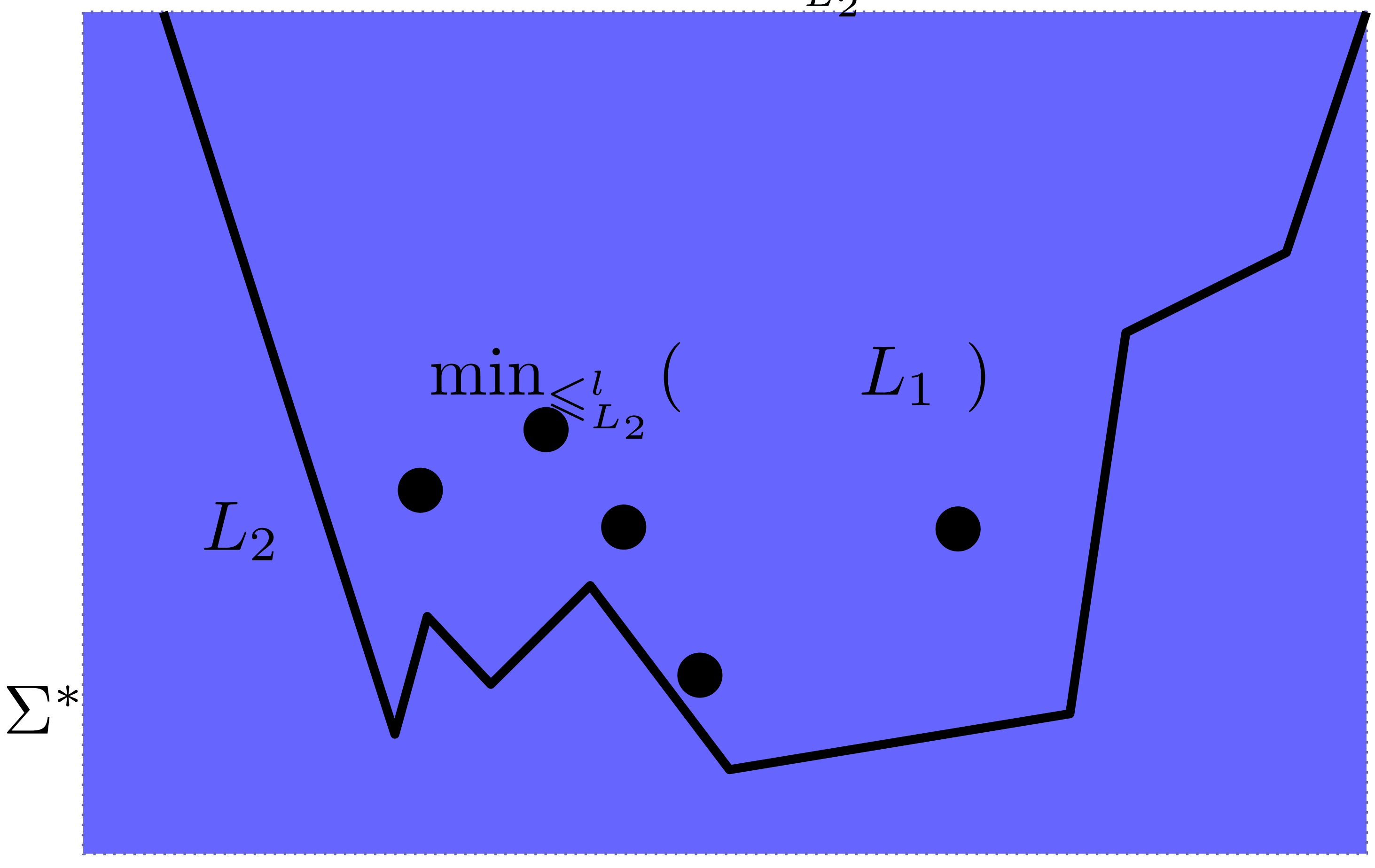

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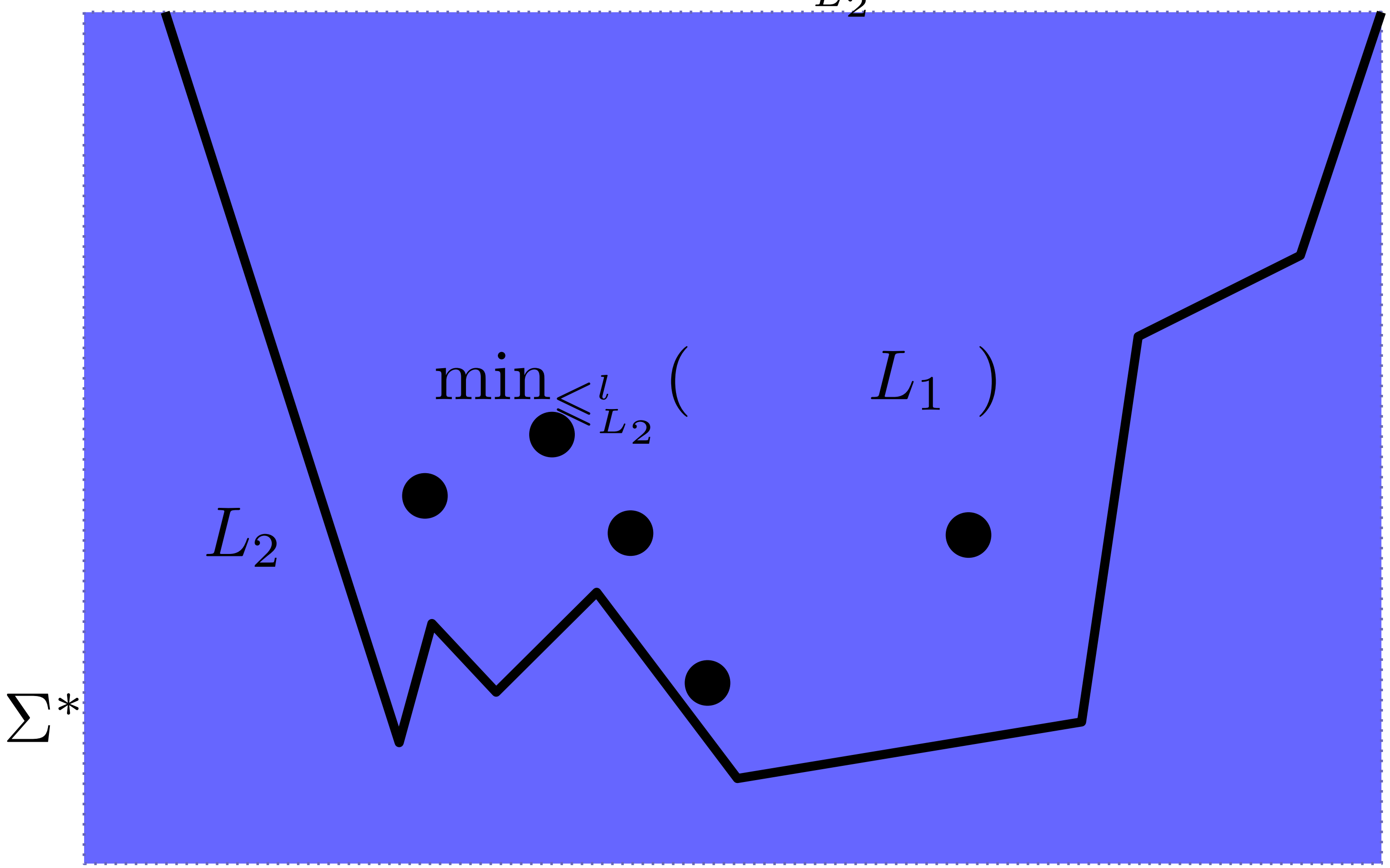

$$L_1 \subseteq L_2 \text{ iff } \forall w \in \min_{\leqslant^l_{L_2}}(L_1)\colon w \in L_2$$

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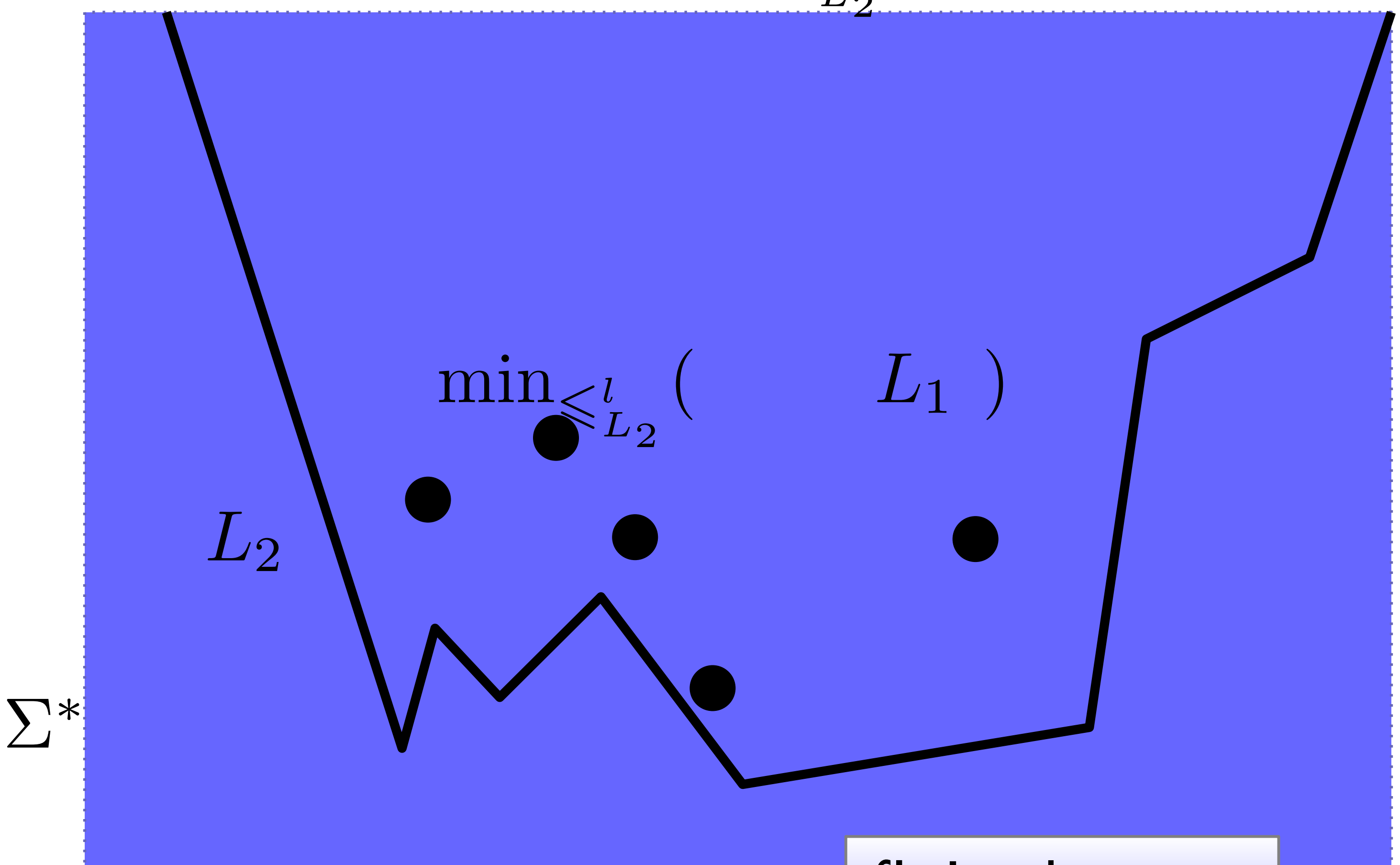

finite by wqo

$L_1 \subseteq L_2$ iff $\forall w \in \min_{\leqslant^l_{L_2}}(L_1)\colon w \in L_2$

Computing $\min_{\leqslant^l_{L_2}}(L_1)$

$$\min_{\leqslant^l_{L_2}}(L_1)$$

$$=$$

$$\min_{\leqslant^l_{L_2}}\left(\text{lfp}\,\lambda X.\,(b \cup Fn(X))\right)$$

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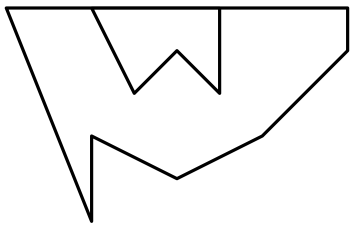

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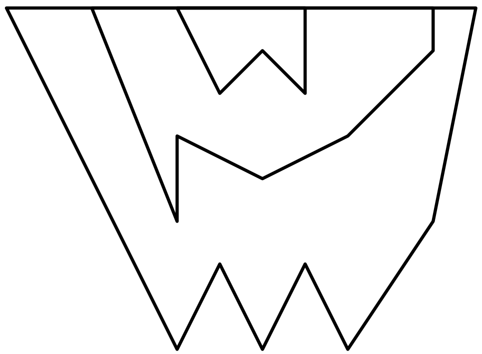

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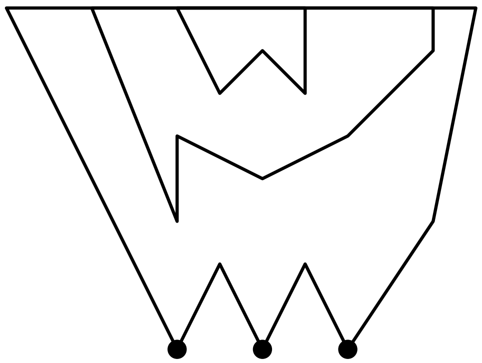

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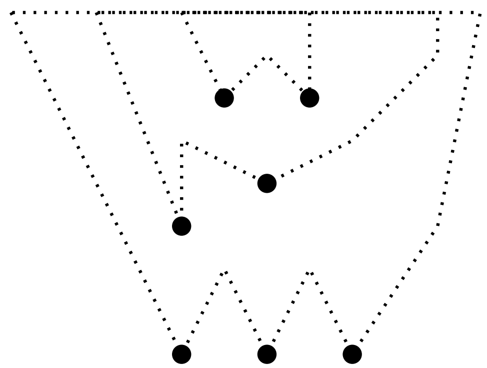

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Word-based antichain algorithm for $\mathcal{L}(\mathcal{A}) \subseteq L_2$

```
Data: FA 𝒜 = ⟨Q, δ, q⁰, F⟩
Data: L₂ regular
⟨Y_q⟩_{q∈Q} := ∅⃗;
repeat
    ⟨X_q⟩_{q∈Q} := ⟨Y_q⟩_{q∈Q};
    ⟨Y_q⟩_{q∈Q} := min_{⩽^l_{L₂}} (b ∪ Fn(⟨Y_q⟩_{q∈Q}));
until ρ_{⩽^l_{L₂}}(⟨Y_q⟩_{q∈Q}) ⊆ ρ_{⩽^l_{L₂}}(⟨X_q⟩_{q∈Q});
forall u ∈ Y_{q⁰} do
    if u ∉ L₂ then return false;
return true;
```

What else?

Nerode left quasiorder relative to L_2

[Varricchio,deLuca'94]

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State based quasiorder

Let $\mathcal{A}_2 = \langle Q_2, \delta_2, q_2^0, F_2 \rangle$ be an automaton for L_2

$$x \leqslant^{l}_{\mathcal{A}_2} y \Leftrightarrow \mathrm{pre}_x(F_2) \subseteq \mathrm{pre}_y(F_2)$$

What else?

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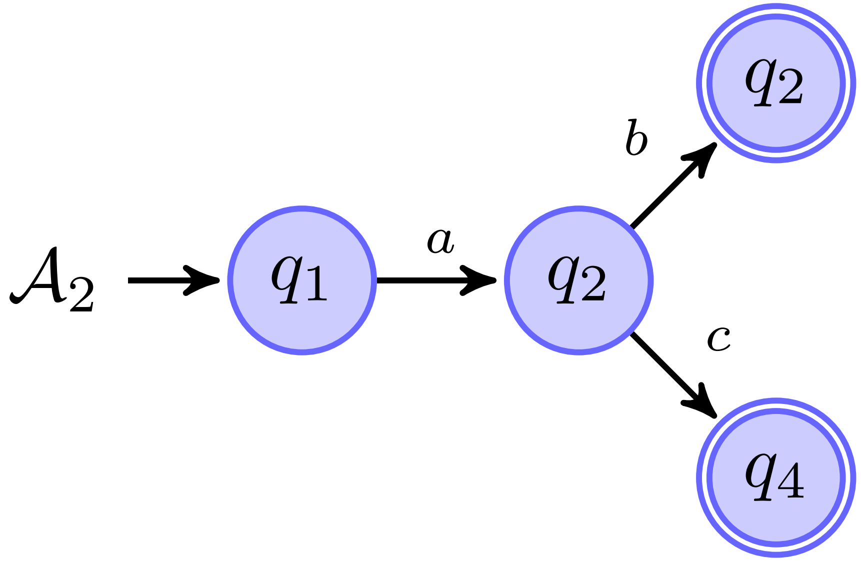

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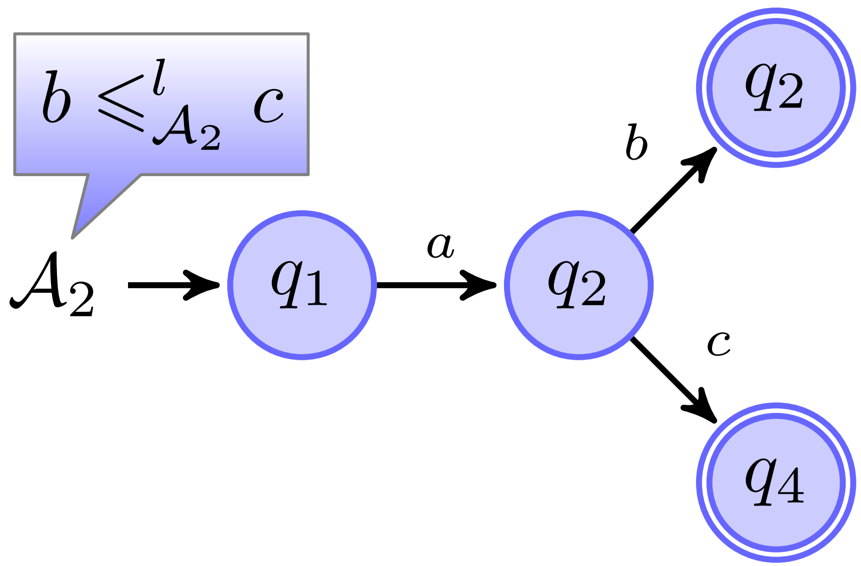

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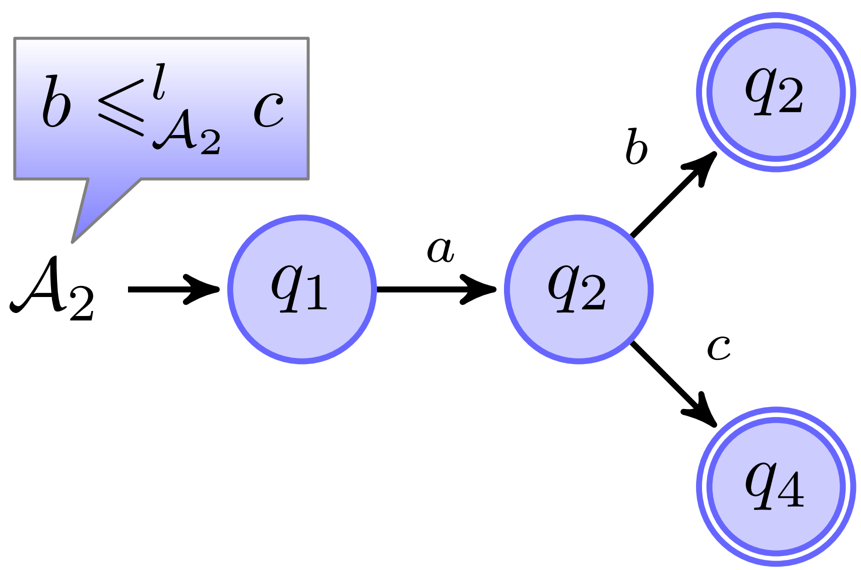

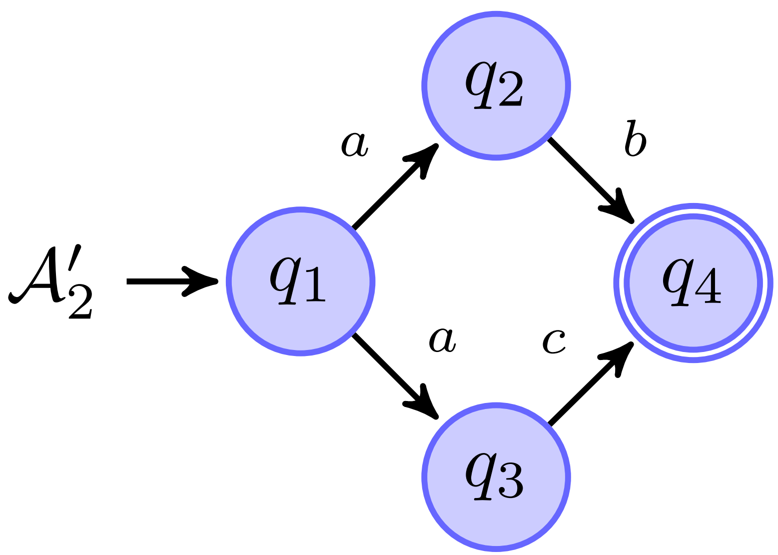

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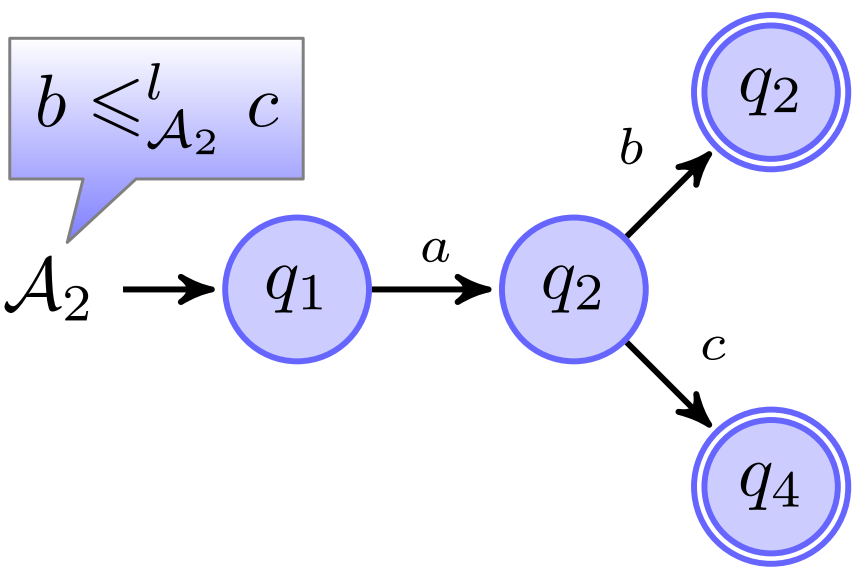

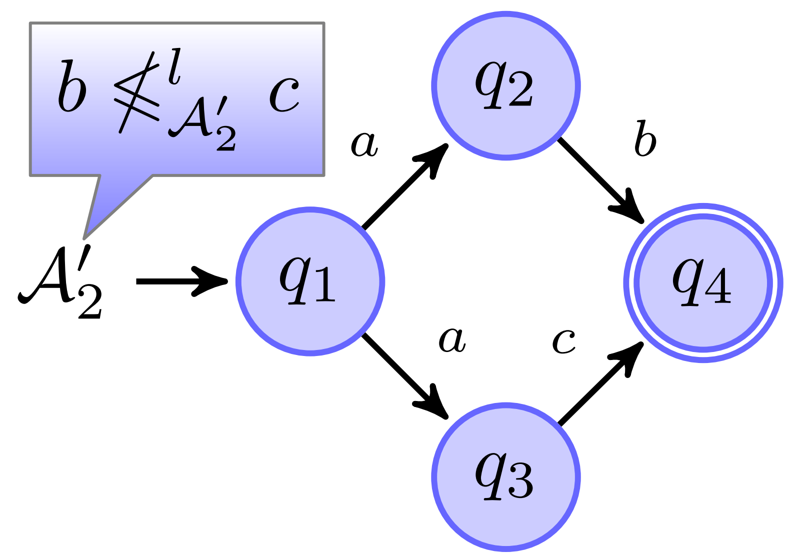

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$\leqslant^l_{\mathcal{A}_2}$ is left L_2 consistent decidable wqo

Word-based antichain algorithm for $\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$

```
Data: FA A_1 = ⟨Q_1, δ_1, q_1^0, F_1⟩
Data: FA A_2 = ⟨Q_2, δ_2, q_2^0, F_2⟩
⟨Y_q⟩_{q∈Q_1} := ∅⃗;
repeat
    ⟨X_q⟩_{q∈Q_1} := ⟨Y_q⟩_{q∈Q_1};
    ⟨Y_q⟩_{q∈Q_1} := min_{⩽^l_{A_2}} (b ∪ Fn(⟨Y_q⟩_{q∈Q_1}));
until ρ_{⩽^l_{A_2}}(⟨Y_q⟩_{q∈Q_1}) ⊆ ρ_{⩽^l_{A_2}}(⟨X_q⟩_{q∈Q_1});
forall u ∈ Y_{q_1^0} do
    if u ∉ L(A_2) then return false;
return true;
```

Data: FA $\mathcal{A}_1 = \langle Q_1, \delta_1, q_1^0, F_1 \rangle$
Data: FA $\mathcal{A}_2 = \langle Q_2, \delta_2, q_2^0, F_2 \rangle$
$\langle Y_q \rangle_{q \in Q_1} := \vec{\varnothing}$;
repeat
    $\langle X_q \rangle_{q \in Q_1} := \langle Y_q \rangle_{q \in Q_1}$;
    $\langle Y_q \rangle_{q \in Q_1} := \min_{\leqslant^l_{\mathcal{A}_2}} \big(b \cup Fn(\langle Y_q \rangle_{q \in Q_1}) \big)$;
until $\rho_{\leqslant^l_{\mathcal{A}_2}}(\langle Y_q \rangle_{q \in Q_1}) \subseteq \rho_{\leqslant^l_{\mathcal{A}_2}}(\langle X_q \rangle_{q \in Q_1})$;
forall $u \in Y_{q_1^0}$ **do**
    if $u \notin L(\mathcal{A}_2)$ **then return** *false*;
return *true*;

Word-based antichain algorithm for $\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$

Data: FA $\mathcal{A}_1 = \langle Q_1, \delta_1, q_1^0, F_1 \rangle$
Data: FA $\mathcal{A}_2 = \langle Q_2, \delta_2, q_2^0, F_2 \rangle$

$\langle Y_q \rangle_{q \in Q_1} := \vec{\varnothing}$;

State based quasiorder

Let $\mathcal{A}_2 = \langle Q_2, \delta_2, q_2^0, F_2 \rangle$ be an automaton for L_2

$$x \leqslant^{l}_{\mathcal{A}_2} y \Leftrightarrow \mathrm{pre}_x(F_2) \subseteq \mathrm{pre}_y(F_2)$$

if $u \notin L(\mathcal{A}_2)$ **then return** *false*;

return *true*;

Ditching words altogether

State-based antichain algorithm for $\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$

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Data: FA $\mathcal{A}_2 = \langle Q_2, \delta_2, q_2^0, F_2 \rangle$

$\langle Y_q \rangle_{q \in Q_1} := \vec{\varnothing}$;
repeat
    $\langle X_q \rangle_{q \in Q_1} := \langle Y_q \rangle_{q \in Q_1}$;
    $\langle Y_q \rangle_{q \in Q_1} := [\ldots]$;
until $Y_q \subseteq^{\forall\exists} X_q$, *for all* $q \in Q_1$;
forall $s \in Y_{q_1^0}$ **do**
    if $q_2^0 \notin s$ **then return** *false*;
return *true*;

$$[\ldots] = \min_{\subseteq^{\forall\exists}} \{ \mathrm{pre}_a^{\mathcal{A}_2}(s) \mid \exists a \in \Sigma,\ q' \in \delta_1(q, a), s \in X_{q'} \} \cup F_2$$

- Same as antichain algo [1]
- Derived from general solution instantiated to $\leqslant^{l}_{\mathcal{A}_2}$
- Variants of the antichain algorithm as instantiations

[1] De Wulf, Doyen, Henzinger, and Raskin. *"Antichains: A New Algorithm for Checking Universality of Finite Automata."* In CAV'06. (cited 171 times)

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- Simulation enhanced antichain algo with simulation enhanced qo [2*]

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[2] Abdulla, Chen, Holík, Mayr, and Vojnar. *"When Simulation Meets Antichains."* In TACAS'10.
Bonchi, and Pous. *"Checking NFA Equivalence with Bisimulations up to Congruence."* In POPL'13.

$Reg \subseteq Reg$

1. $\leqslant^{l}_{L_2} \cap (L_2 \times \overline{L_2}) = \varnothing$
2. $x \leqslant^{l}_{L_2} y$ implies $a\,x \leqslant^{l}_{L_2} a\,y$ for all a
3. $\leqslant^{l}_{L_2}$ is a well-quasiorder
4. $\leqslant^{l}_{L_2}$ is decidable

(3 and 4:) if L_2 is regular

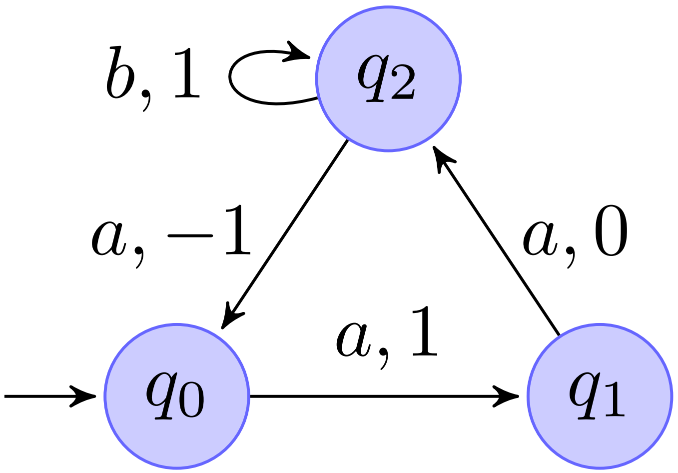
$$L_1 \subseteq L_2 \text{ iff } \operatorname{lfp} \lambda X.\, \min_{\leqslant^{l}_{L_2}} \big(b \cup Fn(X)\big) \subseteq L_2$$

if L_2 is regular then $\leqslant^{l}_{L_2}$ is a wqo

but the opposite might not hold

$Reg \subseteq BeyondRegular$

L_2 is a one-counter net trace set



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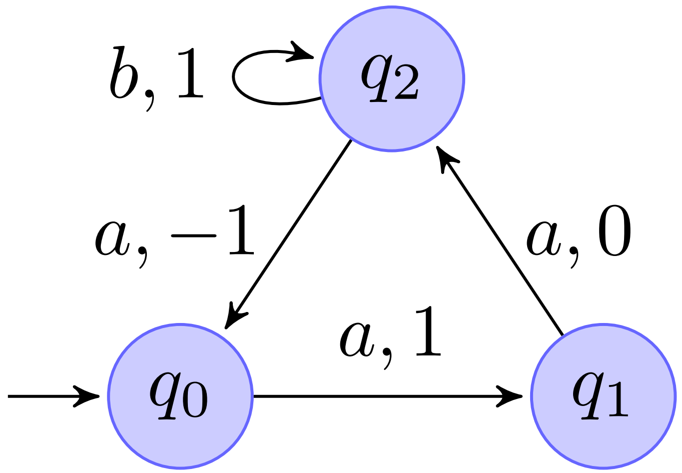

$L_2 = \mathcal{L}(\mathcal{O}_2)$

Traces from configuration

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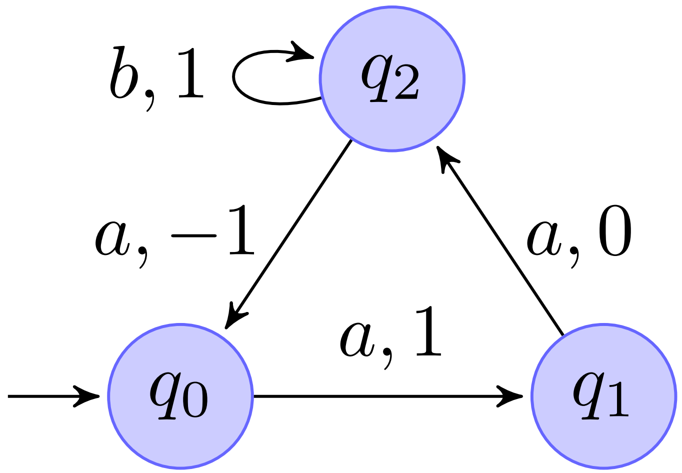

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The right Nerode qo for L_2 ($\leqslant^r_{L_2}$) is a wqo but it's undecidable

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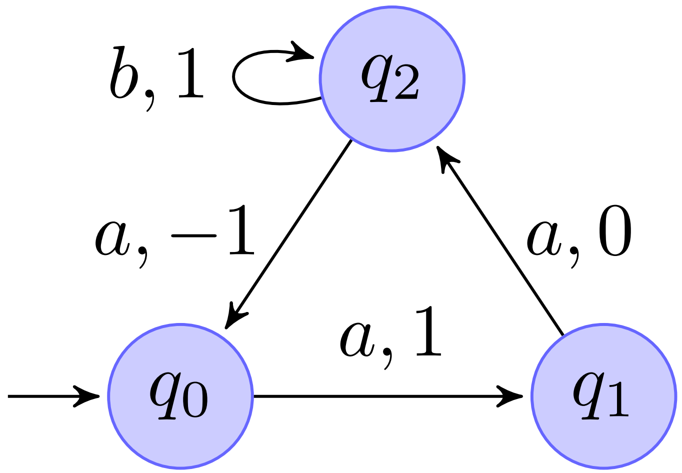

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$u \leqslant^r_{\mathcal{O}_2} v$ iff $m_u \leq m_v$ where $m_u, m_v \in (\mathbb{N} \cup \{\perp\})^3$

$u \leqslant^r_{\mathcal{O}_2} v$ is a right L_2 consistent decidable wqo

Decision procedure for $\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{O}_2)$ using $\leqslant^r_{\mathcal{O}_2}$

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Data: FA $\mathcal{A}_1 = \langle Q, \delta, q^0, F \rangle$
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    $\langle Y_q \rangle_{q \in Q} := \min_{\leqslant^r_{\mathcal{O}_2}} \big(b \cup Fn(\langle Y_q \rangle_{q \in Q})\big)$;
until $\rho_{\leqslant^r_{\mathcal{O}_2}}(\langle Y_q \rangle_{q \in Q}) \subseteq \rho_{\leqslant^r_{\mathcal{O}_2}}(\langle X_q \rangle_{q \in Q})$;
forall $u \in Y_{q^0}$ **do**
    if $u \notin L_2$ **then return** *false*;
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Dual approach

$$\mathrm{lfp}\,\lambda X.\, b \cup Fn(X) \subseteq L_2 \quad \text{iff} \quad b \subseteq \mathrm{gfp}\,\lambda X.\, L_2 \cap \widetilde{Fn}(X)$$

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Does always terminate [1]

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new algos

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State based quasi order [Holík, Meyer @ Netys'15]

$$x \leqslant_{\mathcal{A}_2} y \Leftrightarrow \{(q,q') \mid q \overset{x}{\rightsquigarrow} q'\} \subseteq \{(q,q') \mid q \overset{y}{\rightsquigarrow} q'\}$$

Conclusions

Complete Abstract Interpretations for

$Reg \subseteq Reg$ $\quad$ $Reg \subseteq OCN$ $\quad$ $CFL \subseteq Reg$

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Systematic derivation of algorithms

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- ▷ New perspective on antichains algorithms

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Technical Details

- ▷ SAS 2019
- ▷ arxiv 1904.01388

Future Work

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▷ reduction $\mathrm{gfp} \mapsto \mathrm{lfp}$ using ultimately periodic words

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Regular tree languages

- On the MyHill-Nerode Theorem for Trees by Kozen

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Regular tree languages

$\triangleright$ On the MyHill-Nerode Theorem for Trees by Kozen

Timed Languages (TA $\subseteq$ 1-clock TA)

$\triangleright$ decidability by Ouaknine, Worrell (LICS 2004); reduction to emptiness of weakly alternating timed automata by Lasota, Walukiewicz (ACM ToCL 2008)