Term Modal Logic

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- Kripke structure: $\mathcal{M} = (\mathcal{W}, \mathcal{R}_1, \ldots R_n, \rho)$ is a structure where: $R_i \subseteq (\mathcal{W} \times \mathcal{W})$ and $\rho : \mathcal{W} \to 2^{\mathcal{P}}$.

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- Semantics:

  $\mathcal{M}, w \models p$ iff $p \in \rho(w)$

  $\mathcal{M}, w \models \Box_i\, \varphi$ iff for every $u \in \mathcal{W}$ if $(w, u) \in \mathcal{R}_i$ then $\mathcal{M}, u \models \varphi$.

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- $\forall x \Diamond_x p \ \wedge \ \exists y \Box_y p$
- $\forall x \forall y \ (\neg \Box_x \Box_y p \wedge \Box_x (\exists y \Diamond_y \neg p))$

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  $\forall x\big(A(x) \supset \Diamond_x(\exists y\ B(y) \wedge Syn(x,y))\big)$.

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  $\forall x\big(A(x) \supset \Diamond_x(\exists y\ B(y) \wedge Syn(x,y)\big)\big)$.
- For every client there is some server which will handle the request.
  $\forall x\big(C(x) \supset \exists y\ \big(S(y) \wedge \Box_y(\neg C(x)\big)\big)\big)$

Syntax

Given a set countable of propositions $\mathcal{P}$ and a countable set of variables $\mathcal{V}$ the syntax of term modal logic is defined by

$$\varphi \ := \ p \mid x = y \mid \neg\varphi \mid (\varphi \land \varphi) \mid \exists x \ \varphi \mid \Diamond_x \ \varphi$$

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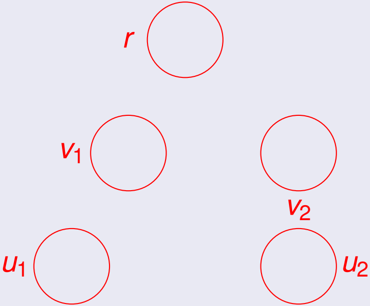
- No constants, No function symbols

Structure

A TML model is given by $\mathcal{M} = (\mathcal{W},$

– $\mathcal{W}$ is non-empty set of worlds



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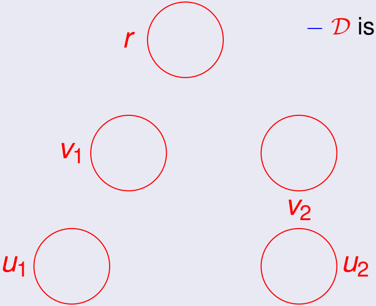

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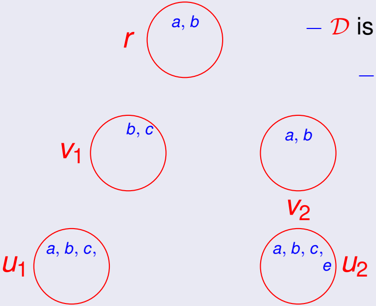
– $\delta : \mathcal{W} \mapsto 2^{\mathcal{D}}$ assigns to each w, a *non-empty* local agent set



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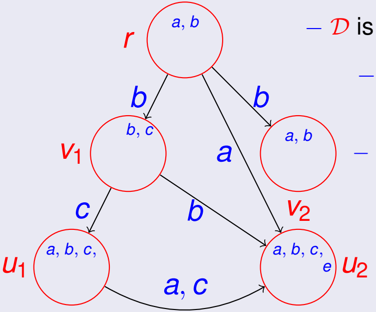

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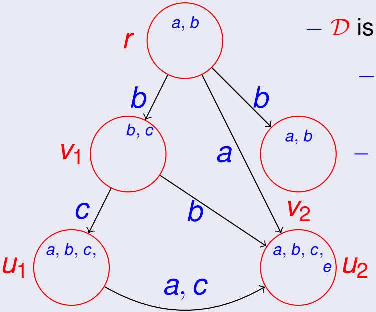

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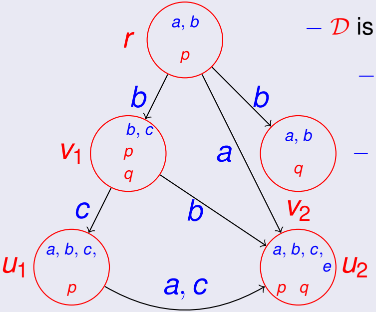

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– $\rho : (\mathcal{W}) \mapsto 2^{\mathcal{P}}$

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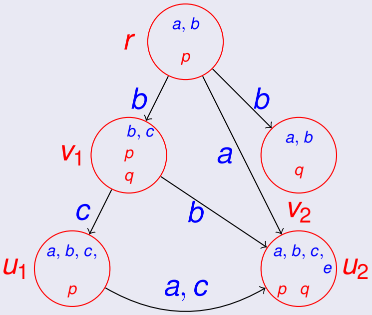

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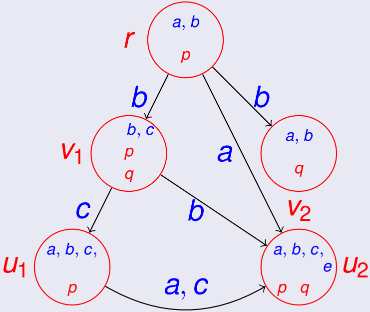

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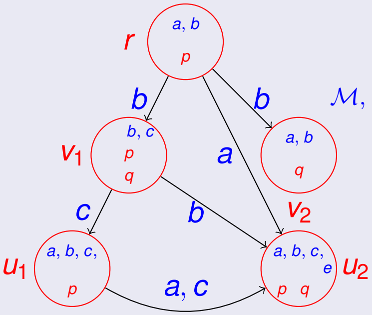

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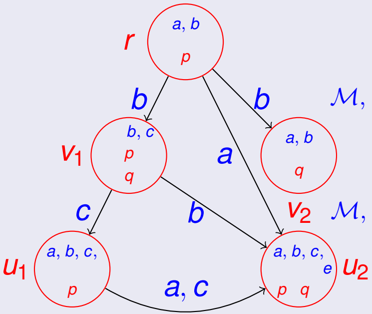

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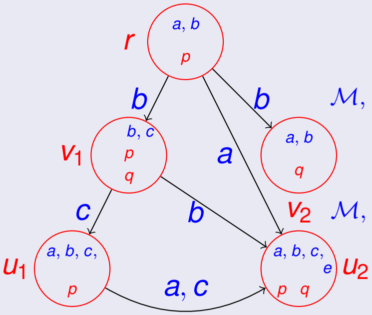

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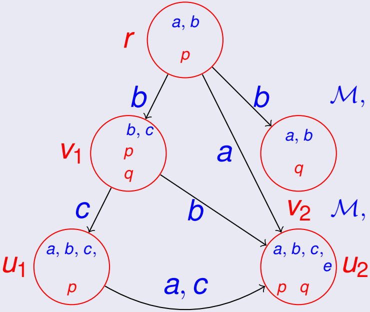

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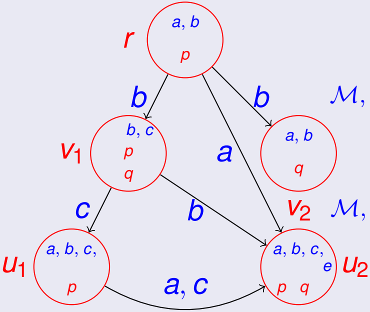

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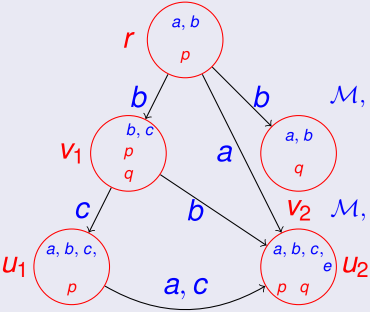

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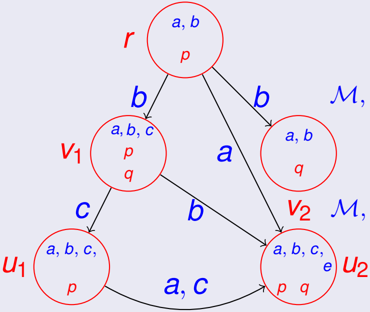

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Any $\varphi \in$ FO is satisfiable iff $Tr_1(\varphi)$ is satisfiable.

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Theorem (Trakhtenbrot type theorem)

For $TML^=$, the FinSat, UnSat and InfAx are mutually recursively inseparable [a].

[a] Sets X and Y are recursively inseparable if there is no Z such that $X \subseteq Z$ and $Z \cap Y = \emptyset$ where checking for membership in Z is decidable.

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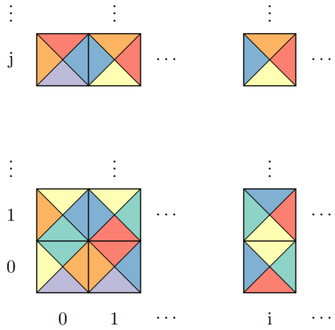
- $\varphi_T \in$ FinSat iff T has some periodic tiling.
- $\varphi_T \in$ InfAx iff T has only aperiodic tiling.
- $\varphi_T \in$ UnSat iff T has no tiling.

Tiling $\mathbb{N} \times \mathbb{N}$



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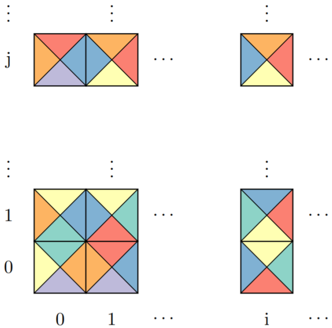

- We induce a discreet linear order with min over agents and use it to construct the grid.

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φ_0	:=	$\exists x \; zero(x)$
φ_{ir}	:=	$\forall x \; \neg \Diamond_x \Diamond_x \top$
φ_{tot}	:=	$\forall x \forall y \; ((x \neq y) \supset (\Diamond_x \Diamond_y \top \vee \Diamond_y \Diamond_x \top))$
φ_{dis}	:=	$\forall x \; (last(x) \vee \exists y \; succ(x, y))$
φ_{trans}	:=	$\forall x \forall y \forall z \; (\Diamond_x \Diamond_y \top \wedge \Diamond_y \Diamond_z \top) \supset (\Diamond_x \Diamond_z \top)$
where,		
$zero(x)$	:=	$\forall y \; \neg \Diamond_y \Diamond_x \top$
$last(x)$	:=	$\forall y \; \neg \Diamond_x \Diamond_y \top$
$succ(x, y)$	:=	$\Diamond_x \Diamond_y \top \; \wedge \forall z \; ((\Diamond_z \Diamond_y \top) \supset (x = z \vee \Diamond_z \Diamond_x \top))$

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Lemma

For every $\mathbb{N}' \subseteq \mathbb{N}$ (either finite or infinite) which is an initial segment of $\mathbb{N}$, there is some $\mathcal{M} = (\mathcal{W}, \mathbb{N}', \delta, \mathcal{R})$ and $r \in \mathcal{W}$ such that $\delta(r) = \mathbb{N}'$ and $\mathcal{M}, r \models \hat{\mathcal{O}}$.

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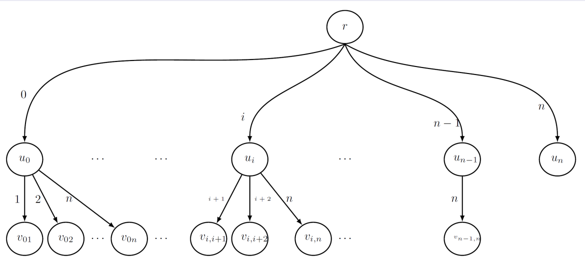
Define $\mathcal{O} = \{\varphi_0, \varphi_{ir}, \varphi_{tot}, \varphi_{dis}\}$ and $\hat{\mathcal{O}} = \bigwedge_{\varphi \in \mathcal{O}} \varphi$.

Lemma

For every $\mathbb{N}' \subseteq \mathbb{N}$ (either finite or infinite) which is an initial segment of $\mathbb{N}$, there is some $\mathcal{M} = (\mathcal{W}, \mathbb{N}', \delta, \mathcal{R})$ and $r \in \mathcal{W}$ such that $\delta(r) = \mathbb{N}'$ and $\mathcal{M}, r \models \hat{\mathcal{O}}$.



Discreet linear order in TML$^=$

Lemma

For any model $\mathcal{M}$ and $r \in \mathcal{W}$, suppose $\mathcal{M}, r \models \hat{\mathcal{O}}$ then let $\mathbb{N}' = [0, \ldots, n-1]$ if $\delta(r)$ is a finite set of size n and $\mathbb{N}' = \mathbb{N}$ otherwise.

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For any model $\mathcal{M}$ *and* $r \in \mathcal{W}$, *suppose* $\mathcal{M}, r \models \hat{\mathcal{O}}$ *then let* $\mathbb{N}' = [0, \ldots, n-1]$ *if* $\delta(r)$ *is a finite set of size* n *and* $\mathbb{N}' = \mathbb{N}$ *otherwise. There exists a function* $f : \mathbb{N}' \to \delta(r)$ *such that for all* $i, j \in \mathbb{N}'$ *if* $i < j$ *then* $\mathcal{M}, w \models \Diamond_{f(i)} \Diamond_{f(j)} \top$.

- Therefore, whenever $\mathcal{M}, r \models \hat{\mathcal{O}}$, wlog there is some initial fragment $\mathbb{N}'$ of $\mathbb{N}$ with $|\mathbb{N}'| = |\delta(r)|$ such that $\mathbb{N}' \subseteq \delta(r)$ and for all $i, j \in \mathbb{N}'$ if $i < j$ then $\mathcal{M}, w \models \Diamond_i \Diamond_j \top$.

Encoding tiling in $TML^=$

- Encode every tile t_i as a path of length i, given by

$$p_i ::= \bigwedge_{j<i} (\forall z \Box_z)^j (\exists z \Diamond_z \top) \land (\forall z \Box_z)^i (\forall z \Box_z \bot)$$

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- (j, k) has tile t_i is encoded as $\Box_j \Box_k p_i$.

$\varphi tile$	$:=$	$\forall z_1 \forall z_2 \forall x \forall y\ \Box_{z_1} \Box_{z_2} ((\Diamond_x \Diamond_y \top) \wedge (\Box_x \Box_y \bigvee_{t_i \in X} p_i))$
$\varphi init$	$:=$	$\forall x\ zero(x) \Rightarrow \forall z_1 \forall z_2 (\Box_{z_1} \Box_{z_2} \Diamond_x \Diamond_x p_0)$
φhor	$:=$	$\forall x \forall y \forall z\ succ(x, y) \Rightarrow$ $(\forall z_1 \forall z_2 \Box_{z_1} \Box_{z_2} (\bigvee_{r_{t_i} = \ell_{t_j}} (\Box_x \Box_z (p_i) \wedge \Box_y \Box_z (p_j)))$
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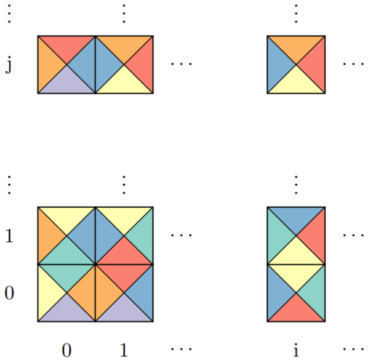

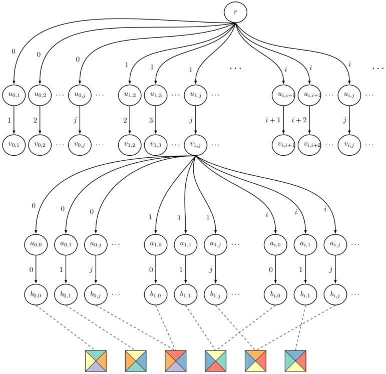

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Any $\varphi \in$ TML is satisfiable iff the corresponding FOML formula $Tr_2(\varphi)$ is satisfiable.

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- Not expressible in TML: (we only have $\Box_x\psi(x, y)$ or $\Box_y\psi(x, y)$. Crucial to get decidability.

PTML^2 decidability

Proof strategy

- Normal form for PTML^2: Combination of Modal normal form and FO^2 normal form

$PTML^2$ decidability

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- Normal form for $PTML^2$: Combination of Modal normal form and FO^2 normal form
- Bounded model property: Induction over FO^2 bounded model construction

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FO types

For a given FO structure $\mathfrak{A} = (\mathcal{D}, \rho)$

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With respect to a given formula, there are at most exponentially many distinct 1-types.

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Bounded model construction

- If $(\mathcal{D}, \rho) \models \forall x \forall y \, \varphi \;\wedge\; \bigwedge_{i=1}^{q} \forall x \exists y \, \psi_i$

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    - $\text{2-type}[(\Lambda, e, f), (\Pi_i, i, f')] = \text{2-type}(a_\Lambda, b_i)$ where $f' = f + 1 \mod 3$

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    - $\text{2-type}[(\Lambda, e, f), (\Pi, e', f')] = \text{2-type}(a_\Lambda, a_\Pi)$ if it is not defined in the previous step.

The type based model continues to satisfy

$$\forall x \forall y\ \varphi\ \wedge\ \bigwedge_{i=1}^{q} \forall x \exists y\ \psi_i$$

Normal form for PTML^2

- Fine normal form for modal logic: DNF where each clause is of the form: $(\bigwedge_i s_i \wedge \Box\alpha \wedge \bigwedge_j \Diamond\beta_j)$ where α, β_j are recursively in the normal form.

Normal form for $PTML^2$

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$$\bigwedge_{i \leq a} s_i \wedge \bigwedge_{z \in \{x,y\}} (\Box_z \alpha \wedge \bigwedge_{j \leq m_z} \Diamond_z \beta_j) \bigwedge_{z \in \{x,y\}} (\forall z\ \gamma \wedge \bigwedge_{k \leq n_z} \exists z\ \delta_k) \wedge \forall x \forall y\ \varphi \wedge \bigwedge_{l \leq b} \forall x \exists y\ \psi_l$$

where s_i are literals and all α, β_j are recursively in the normal form and $\gamma, \delta_k, \varphi, \psi_l$ do not have quantifiers at the outermost level.

PTML^2 type

For any tree model $\mathcal{M}$ rooted at r, (let $\Delta \in \{\Box, \Diamond\}$ and $z \in \{x, y\}$):

- For all $w \in \mathcal{W}$ and for all $c, d \in \delta(w)$, define $\textit{2-type}(w, c, d) = \{\Delta_z \psi(x, y) \mid \mathcal{M}, w \models \Delta_z \psi(c, d)\}$

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$PTML^2$ type

For any tree model $\mathcal{M}$ rooted at r, (let $\Delta \in \{\Box, \Diamond\}$ and $z \in \{x, y\}$):

- For all $w \in \mathcal{W}$ and for all $c, d \in \delta(w)$, define $2\text{-}type(w, c, d) = \{\Delta_z \psi(x, y) \mid \mathcal{M}, w \models \Delta_z \psi(c, d)\}$
- If w is a non root node, (say $w' \xrightarrow{a} w$) then for all $c \in \delta(w)$ define $1\text{-type}(w, c) = (\Lambda_1; \Lambda_2; \Lambda_3)$
  $\Lambda_1 = 2\text{-type}(w, c, c)$; $\Lambda_2 = \{2\text{-type}(w, c, d) \mid d \in \delta(w)\}$;
  $\Lambda_3 = 2\text{-type}(w, c, a)$
- For the root node r, for all $c \in \delta(r)$ define $1\text{-type}(w, c) = (\Lambda_1; \Lambda_2; \{\top\})$

Bounded model construction for PTML^2

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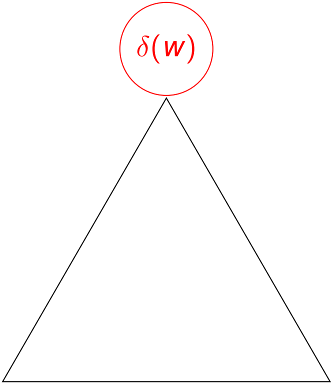
- Let $E_\theta = \{\chi_1, \ldots \chi_q\}$ be the set of all subformulas of θ that need witnesses.

Bounded model construction for $PTML^2$



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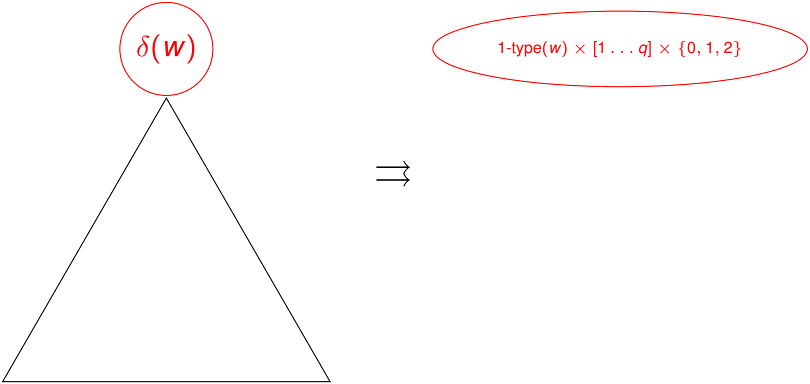
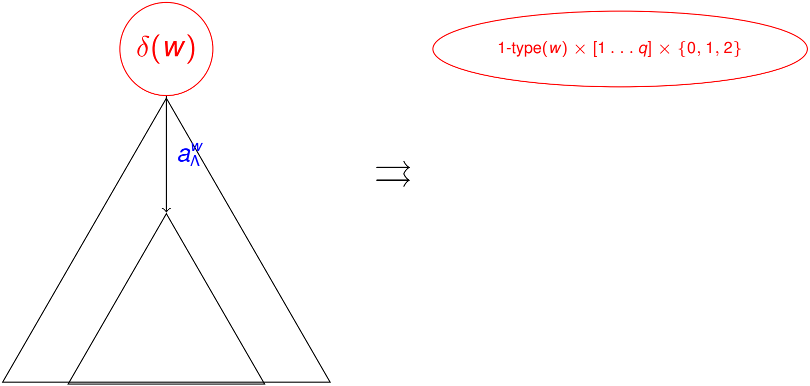


Bounded model construction for PTML^2



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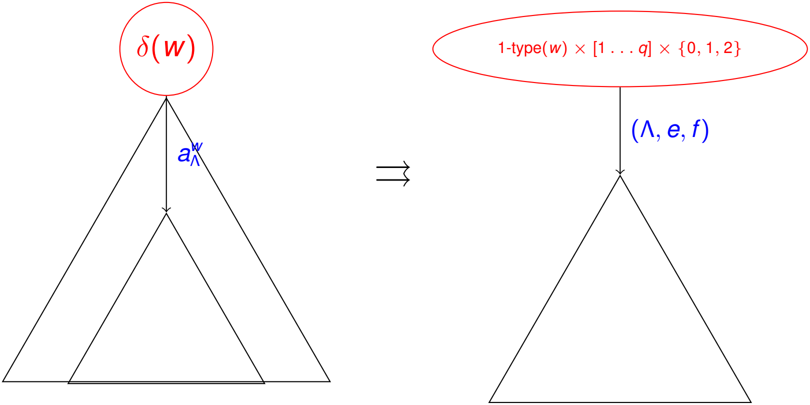

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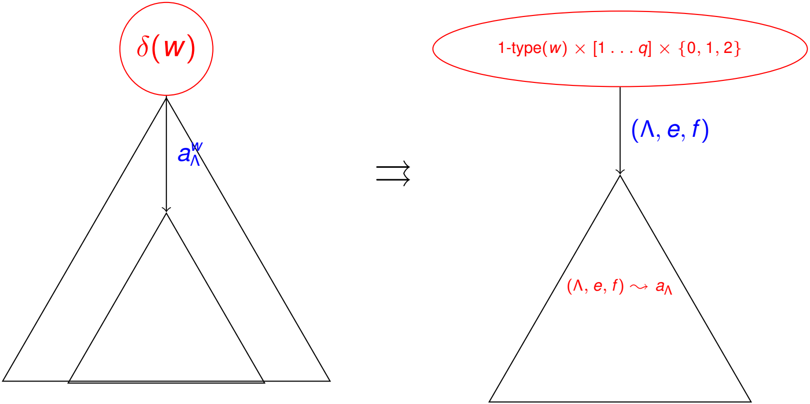

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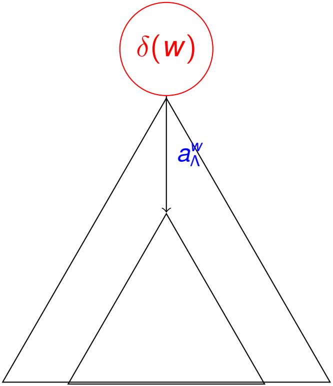
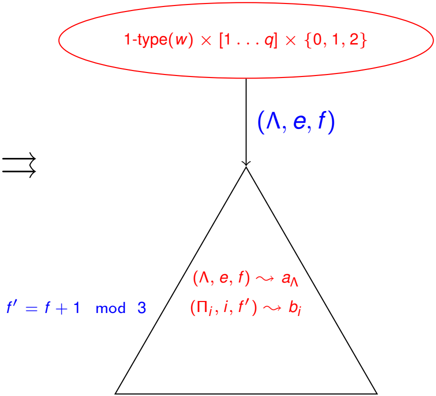


$\Rightarrow$



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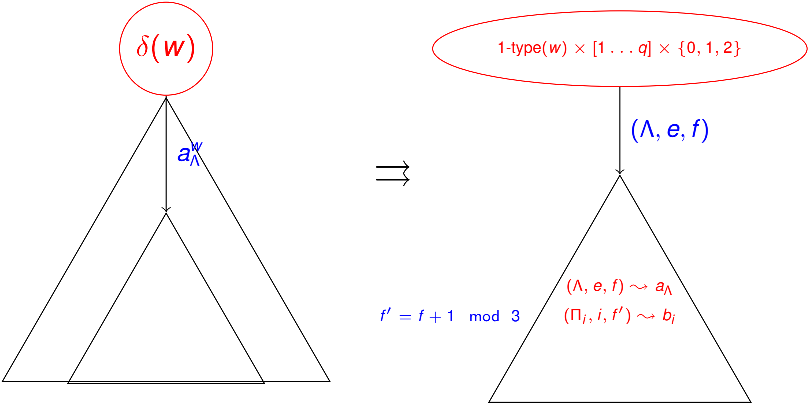

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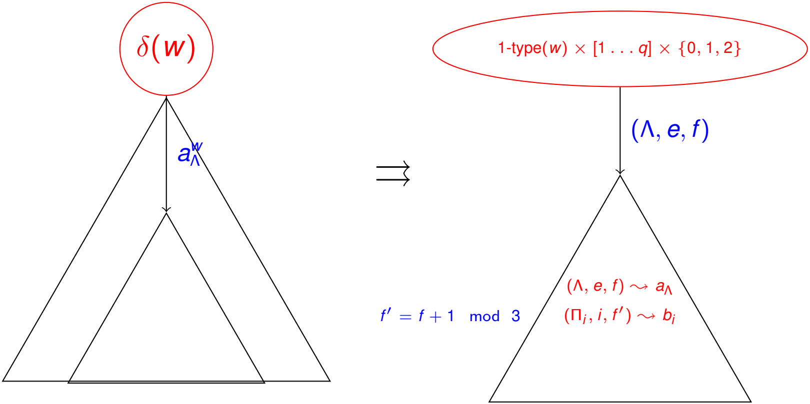

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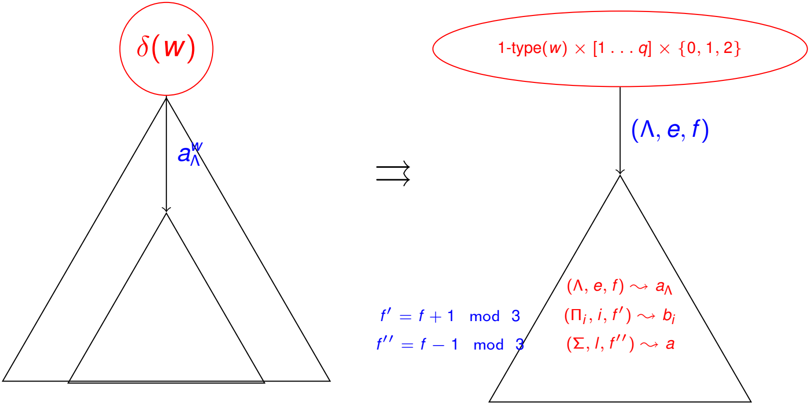

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□

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Model checking

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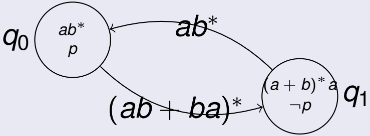
Finite specification



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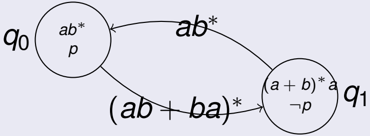
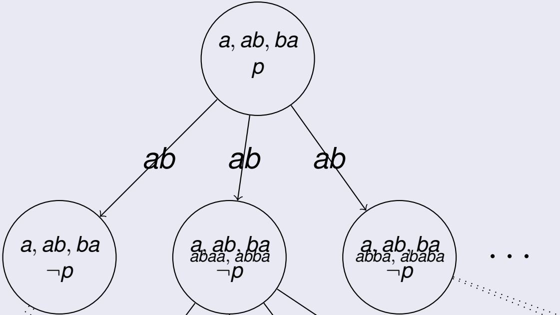


Unravelling



THANK YOU