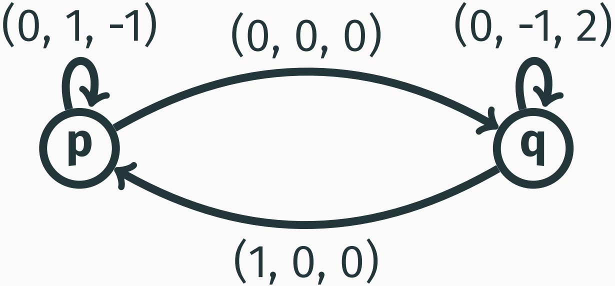
Affine Extensions of Integer Vector Addition Systems with States

Michael Blondin

UNIVERSITÉ DE SHERBROOKE

Joint work with Christoph Haase and Filip Mazowiecki

Vector addition systems with states (VASS)



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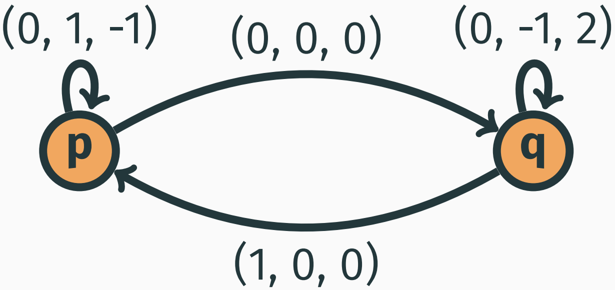
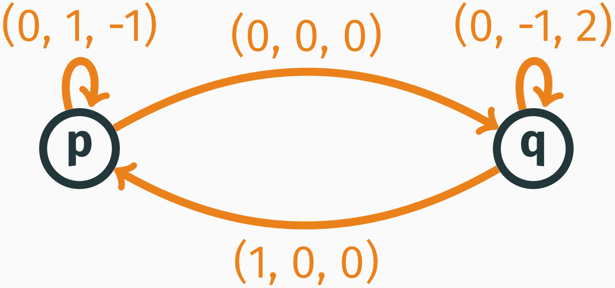


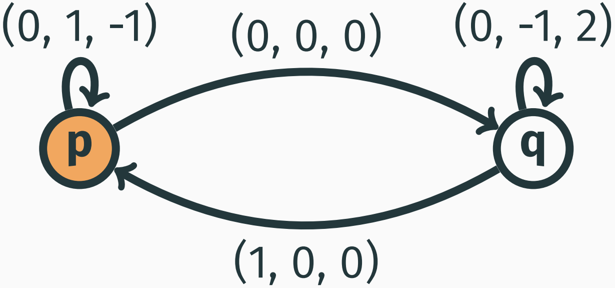
Control-states

Vector addition systems with states (VASS)



Transitions

Vector addition systems with states (VASS)



p(0, 0, 1)

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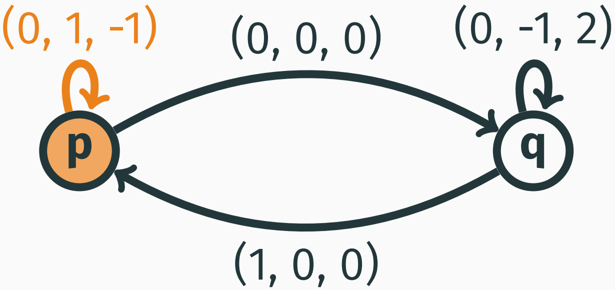

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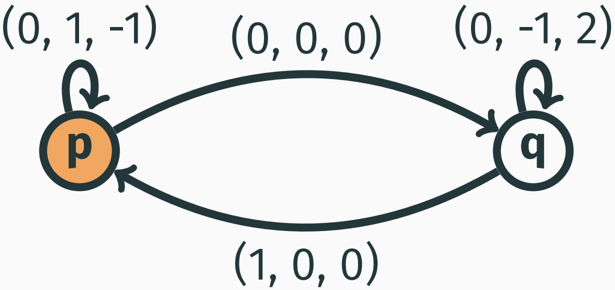

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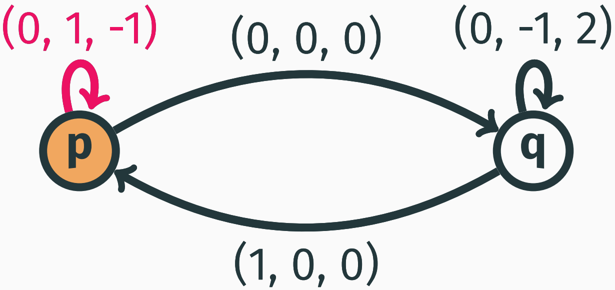

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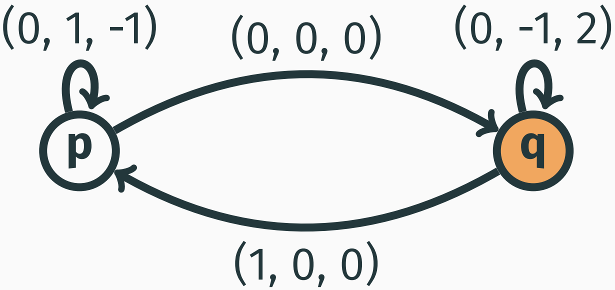

q(0, 1, 0)

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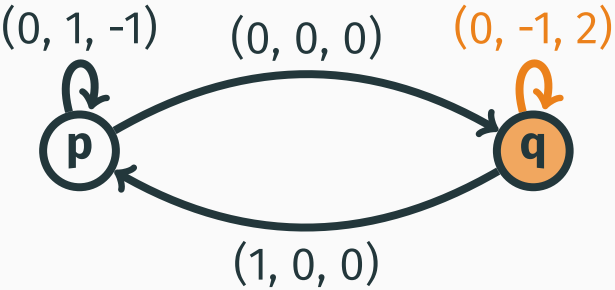

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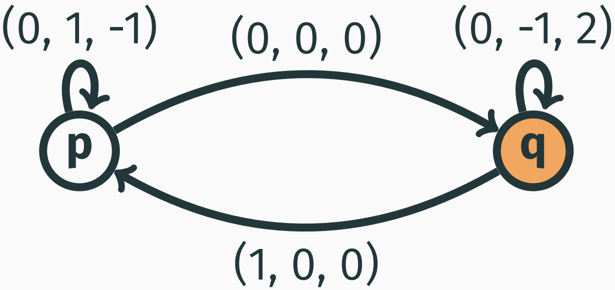

q(0, 0, 2)

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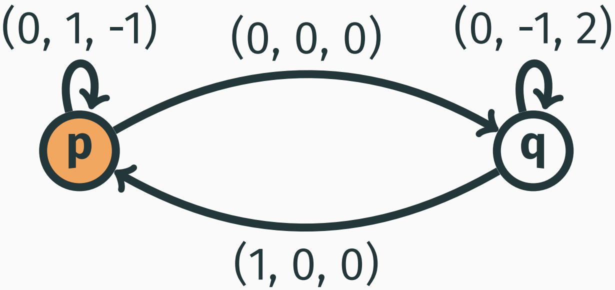
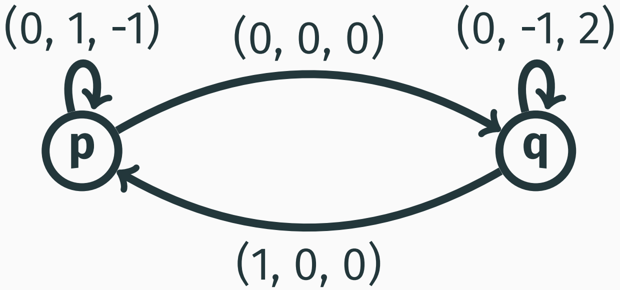


p(1, 0, 2)

Vector addition systems with states (VASS)



$$p(0, 0, 1) \xrightarrow{*}_{\mathbb{N}} p(1, 0, 2)$$

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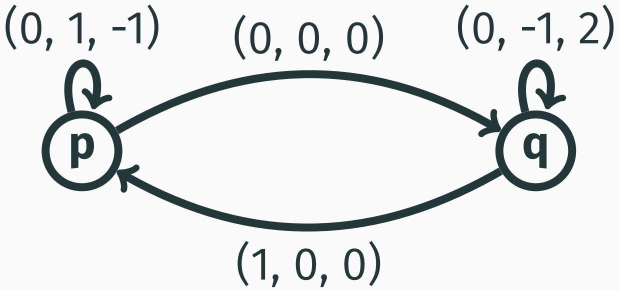
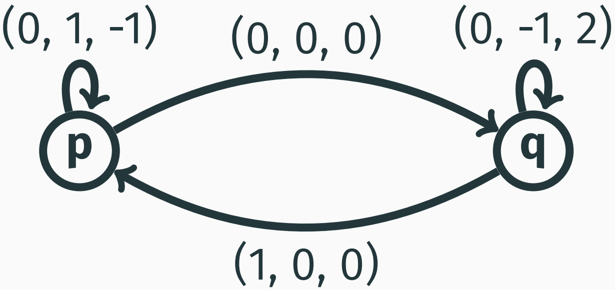


$$\mathrm{p}(0, 0, 1) \xrightarrow{*}_{\mathbb{N}} \mathrm{p}(x, y, z) \iff 0 < y + z \leq 2^x$$

Vector addition systems with states (VASS)



Reachability: $p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{N}} q(\boldsymbol{v})$?

Coverability: $p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{N}} q(\geq \boldsymbol{v})$?

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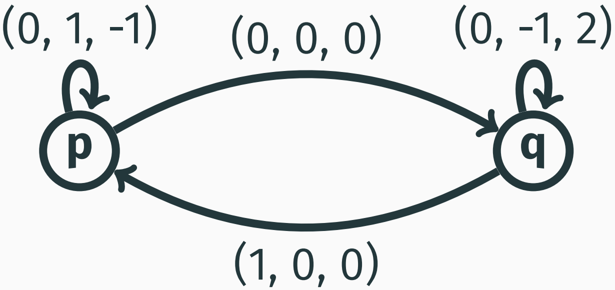

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Vector addition systems with states (VASS)

Concurrent programs

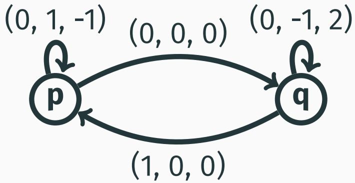
Protocols

Business processes

Biological processes

⋮

correct? →



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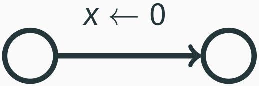
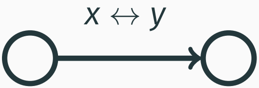
Common operations used for modeling:

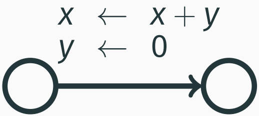
Reset



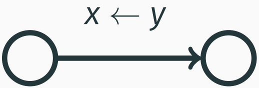
Swap



Transfer



Copy



Vector addition systems with states (VASS)

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$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Swap

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Transfer

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Copy

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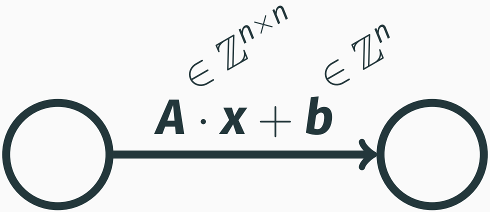
All affine transformations!

Vector addition systems with states (VASS)

Affine VASS:



Complexity of reachability and coverability

	No extensions	+ Resets	+ Transfers
$\xrightarrow{*}_{\mathbb{N}}$	TOWER-hard (CLLLM ’19) $\in$ Ackermann (Leroux, Schmitz ’19)	Undecidable (Araki, Kasami ’76)	
$\xrightarrow{*}_{\mathbb{N}} \geq$	EXPSPACE-complete (Lipton ’76, Rackoff ’78)	Ackermann-complete (Schnoebelen ’02, Figueira *et al.* ’11)	

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Intractable!

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- Can be alleviated by using an over-approximation of $\xrightarrow{*}_{\mathbb{N}}$
- Successful in practice, *e.g.* Esparza *et al.* CAV'14, B. *et al.* TACAS'16, Geffroy *et al.* RP'16, Athanasiou *et al.* IJCAR'16
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$\xrightarrow{*}_{\mathbb{Z}}$ $\xrightarrow{*}_{\mathbb{Z}} \geq$	NP-complete (Haase, Halfon ’14)		?

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$\xrightarrow{*}_{\mathbb{Z}}$ $\xrightarrow{*}_{\mathbb{Z}} \geq$	NP-complete (new proof)		PSPACE-complete

Our contribution

- Any affine $\mathbb{Z}$-VASS with finite matrix monoid can be translated into an equivalent $\mathbb{Z}$-VASS
- Reachability relation of such affine $\mathbb{Z}$-VASS is semilinear
- Classification of complexity w.r.t. extensions

Complexity of reachability and coverability

Related work

- Finkel and Leroux (FSTTCS'12)

  Accelerations of affine counter machines without control-states

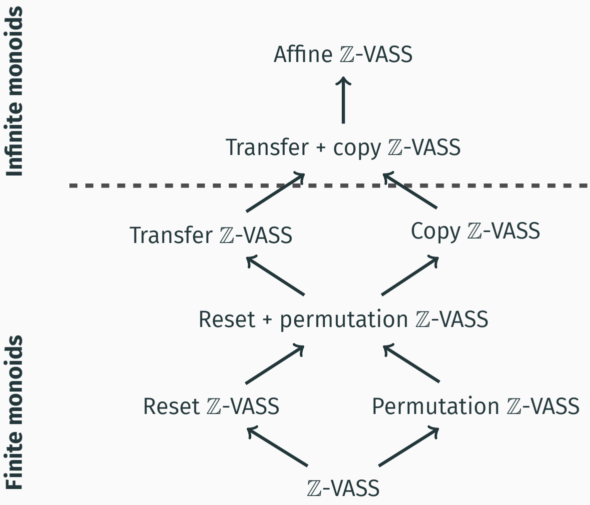
- Iosif and Sangnier (ATVA'16)

  Complexity of model checking over flat structures with guards defined by convex polyhedra

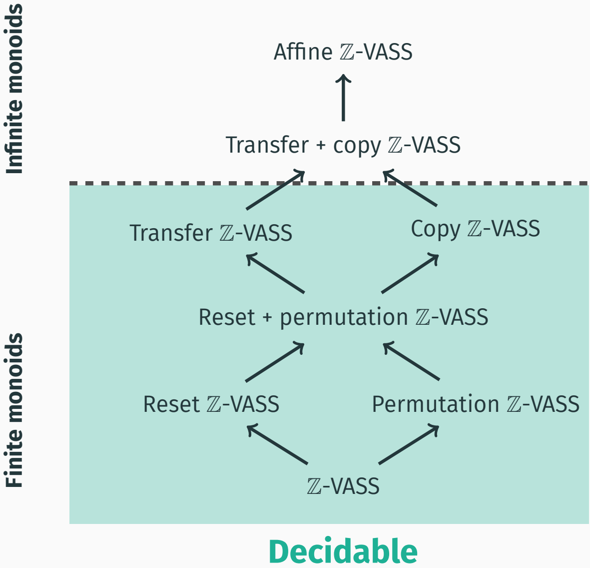
- Cadilhac, Finkel and McKenzie (IJFCS'12)

  Affine Parikh automata with finite-monoid restriction

Overview


Infinite monoids
Finite monoids
Affine $\mathbb{Z}$-VASS
Transfer + copy $\mathbb{Z}$-VASS
Transfer $\mathbb{Z}$-VASS
Copy $\mathbb{Z}$-VASS
Reset + permutation $\mathbb{Z}$-VASS
Reset $\mathbb{Z}$-VASS
Permutation $\mathbb{Z}$-VASS
$\mathbb{Z}$-VASS

Overview



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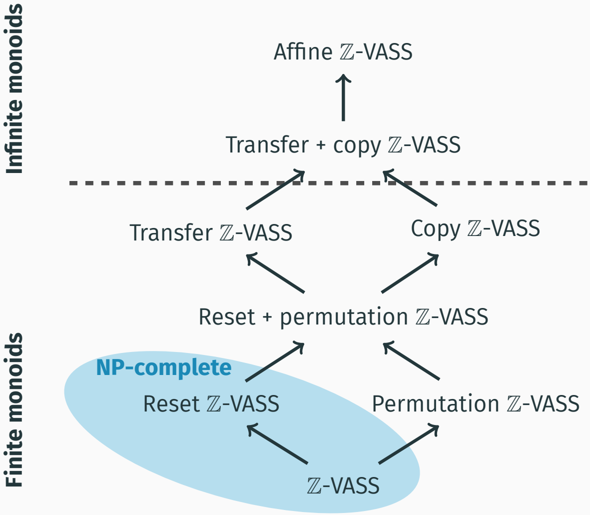
Decidable

Overview



Decidable

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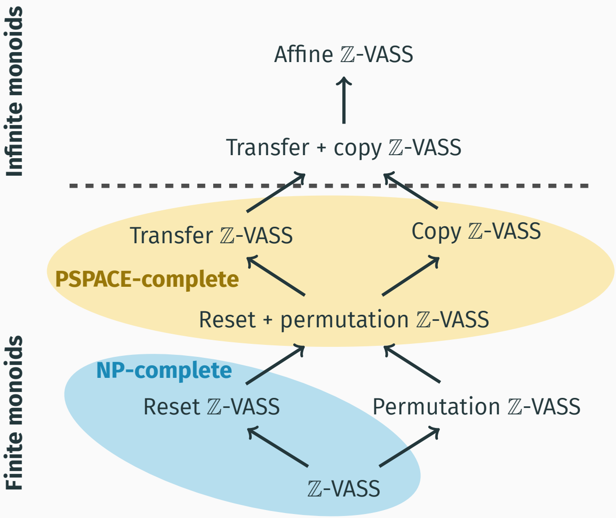

Decidable

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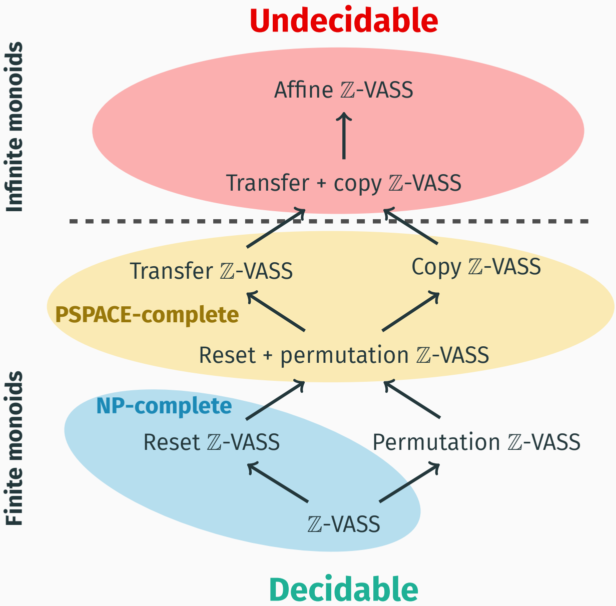
Undecidable
Infinite monoids
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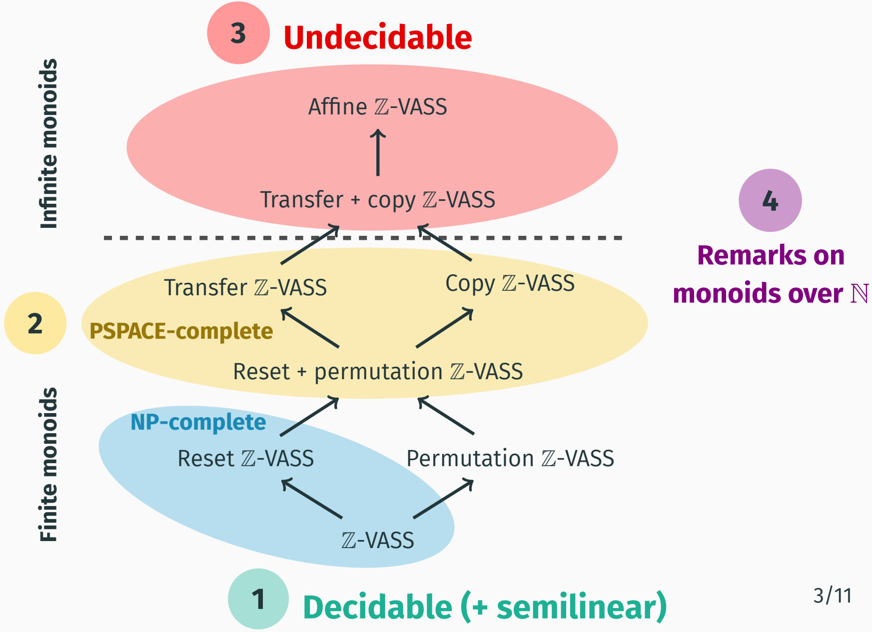
3
Undecidable
Infinite monoids
Affine ℤ-VASS
Transfer + copy ℤ-VASS
4
Remarks on
monoids over ℕ
Transfer ℤ-VASS
Copy ℤ-VASS
2
PSPACE-complete
Reset + permutation ℤ-VASS
Finite monoids
NP-complete
Reset ℤ-VASS
Permutation ℤ-VASS
ℤ-VASS
1
Decidable (+ semilinear)

A few definitions

For every transition $t\colon (p) \xrightarrow{\boldsymbol{A}\cdot\boldsymbol{x}+\boldsymbol{b}} (q)$ and $\sigma \in T^*$, let

$$M_\varepsilon = \boldsymbol{I} \qquad \varepsilon(\boldsymbol{u}) = \boldsymbol{u}$$

$$M_{\sigma t} = \boldsymbol{A} \cdot M_\sigma \qquad \sigma t(\boldsymbol{u}) = \boldsymbol{A} \cdot \sigma(\boldsymbol{u}) + \boldsymbol{b}$$

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Matrix | Effect

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Matrix monoid

$$\mathcal{M}_\mathcal{V} = \{M_w : w \in T^*\}$$

From affine $\mathbb{Z}$-VASS to $\mathbb{Z}$-VASS

Theorem

Let $\mathcal{V}$ be an affine $\mathbb{Z}$-VASS. If $\mathcal{M}_{\mathcal{V}}$ is finite, then $\exists$ $\mathbb{Z}$-VASS $\mathcal{V}'$ s.t.

- $p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{Z}} q(\boldsymbol{v})$ *in* $\mathcal{V} \iff p(\boldsymbol{u}, \boldsymbol{0}) \xrightarrow{*}_{\mathbb{Z}} q(\boldsymbol{0}, \boldsymbol{v})$ *in* $\mathcal{V}'$
- $|\mathcal{V}'| \in \mathrm{poly}(|\mathcal{V}|, |\mathcal{M}_{\mathcal{V}}|)$

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Proof sketch

$p(\boldsymbol{u}) \xrightarrow{w}_{\mathbb{Z}} q(\boldsymbol{v}) \iff$

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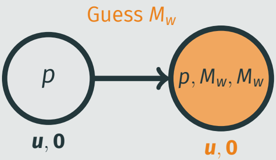

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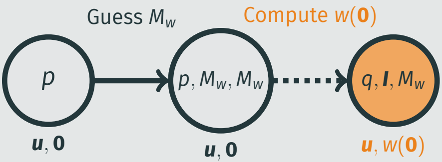

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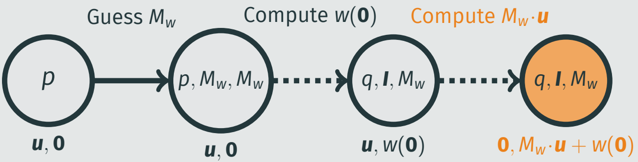

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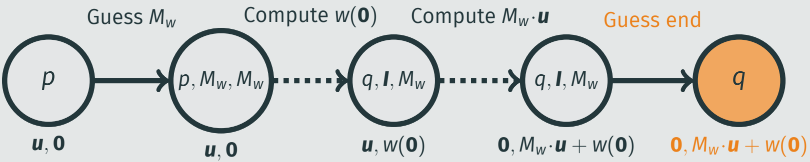

From affine $\mathbb{Z}$-VASS to $\mathbb{Z}$-VASS

Theorem

Let $\mathcal{V}$ be an affine $\mathbb{Z}$-VASS. If $\mathcal{M}_\mathcal{V}$ is finite, then $\exists$ $\mathbb{Z}$-VASS $\mathcal{V}'$ s.t.

- $p(\boldsymbol{u}) \xrightarrow{*}_\mathbb{Z} q(\boldsymbol{v})$ *in* $\mathcal{V} \iff p(\boldsymbol{u}, \boldsymbol{0}) \xrightarrow{*}_\mathbb{Z} q(\boldsymbol{0}, \boldsymbol{v})$ *in* $\mathcal{V}'$
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Proof sketch

$p(\boldsymbol{u}) \xrightarrow{w}_\mathbb{Z} q(\boldsymbol{v}) \iff$

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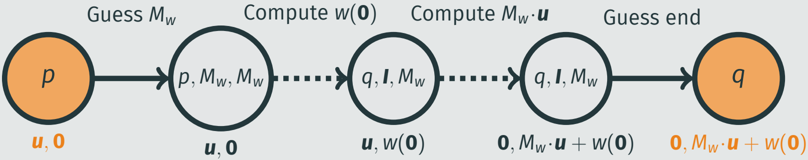

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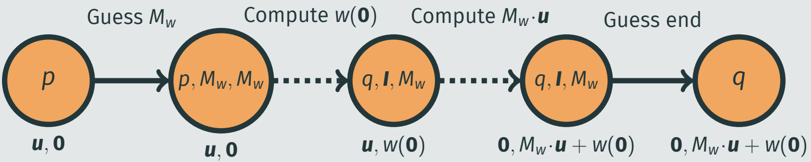

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Corollary

Reachability is decidable for

affine $\mathbb{Z}$-VASS with finite matrix monoid

Semilinearity of affine $\mathbb{Z}$-VASS

Corollary

If an affine $\mathbb{Z}$-VASS has a finite monoid, then

$$\left\{(\boldsymbol{u}, \boldsymbol{v}) : p(\boldsymbol{u}) \xrightarrow{*}_{\mathbb{Z}} q(\boldsymbol{v})\right\} \text{ is semilinear}$$

Proof

Follows from our translation and

known result on $\mathbb{Z}$-VASS (Haase, Halfon RP'14)

Semilinearity of affine $\mathbb{Z}$-VASS

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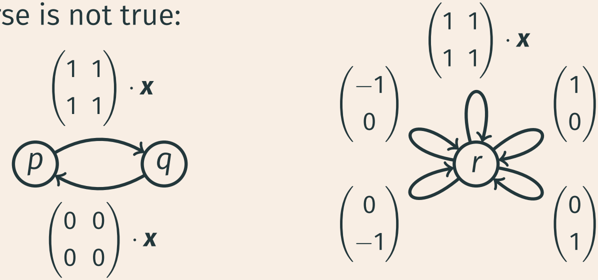
Observation

Converse is not true:



Semilinearity of affine $\mathbb{Z}$-VASS

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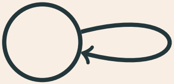
Observation **Boigelot '98, Finkel and Leroux '02**

Converse is true for single state and single transition:



$\boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{b}$

Reachability in transfer $\mathbb{Z}$-VASS is in PSPACE

- Transfer matrix: exactly one 1 per column, hence $|\mathcal{M}_{\mathcal{V}}| \leq n^n = 2^{n \log n}$
- Transform transfer $\mathbb{Z}$-VASS $\mathcal{V}$ into $\mathbb{Z}$-VASS $\mathcal{V}'$ of size $\mathrm{poly}(|\mathcal{V}|, 2^{n \log n})$
- $\mathbb{Z}$-reachability has witnesses of the form $w_1^{k_1} w_2^{k_2} \cdots w_\ell^{k_\ell}$ where $|w_1 w_2 \cdots w_\ell| \leq \mathrm{poly}(|\mathcal{V}'|)$ (B. *et al.* LICS'15)
- Guess witness on the fly with polynomial space

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Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

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$$q_0\,(\overset{a}{1}\ \overset{b}{0}\ \overset{a}{0}\ \overset{b}{1}\ \cdots\ \overset{a}{0}\ \overset{b}{1})$$

Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

$$
\begin{matrix} & a & b & a & b & & a & b & \\ q_0 (& 1 & 0 & 0 & 1 & \cdots & 0 & 1 &) \end{matrix}
$$

Reachability in transfer $\mathbb{Z}$-VASS is PSPACE-hard

Idea: simulate linear bounded Turing machine

$$r_1 \begin{pmatrix} \overset{a}{1} & \overset{b}{0} & \overset{a}{0} & \overset{b}{1} & \cdots & \overset{a}{0} & \overset{b}{1} \end{pmatrix}$$

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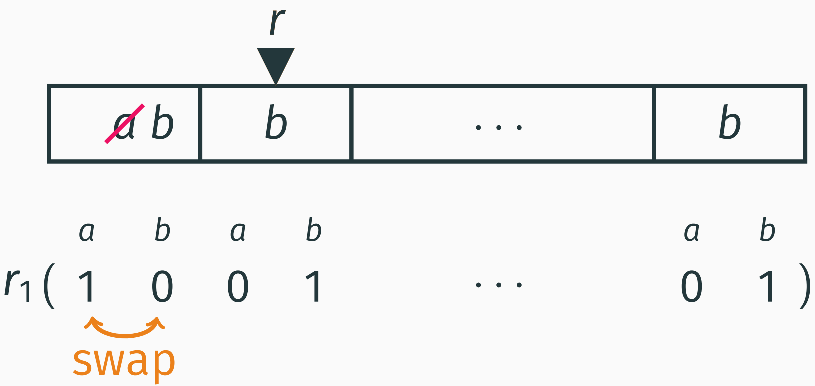

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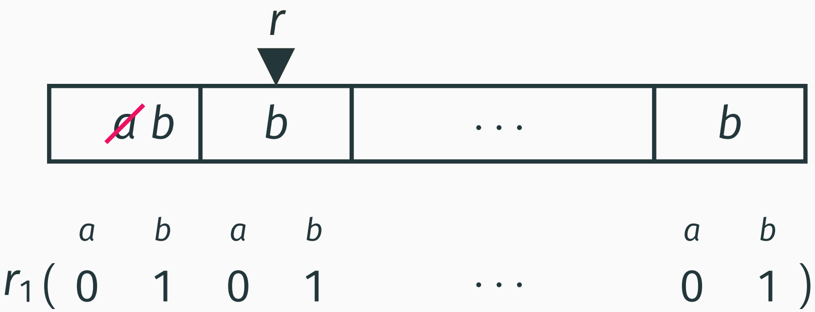

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reset

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Idea: simulate linear bounded Turing machine

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Simulation is faithful iff
the sum of bits is left unchanged

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Idea: simulate linear bounded Turing machine

$$r_1\begin{pmatrix} \overset{a}{0} & \overset{b}{1} & \overset{a}{0} & \overset{b}{1} & \cdots & \overset{a}{0} & \overset{b}{1} \end{pmatrix}$$

Swaps and resets

can be simulated by transfers

Reachability in affine $\mathbb{Z}$-VASS is undecidable

Reduction from the Post correspondence problem

$$w_1 = \frac{10}{1} \qquad w_2 = \frac{01}{0} \qquad w_3 = \frac{1}{011}$$

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Reichert '15

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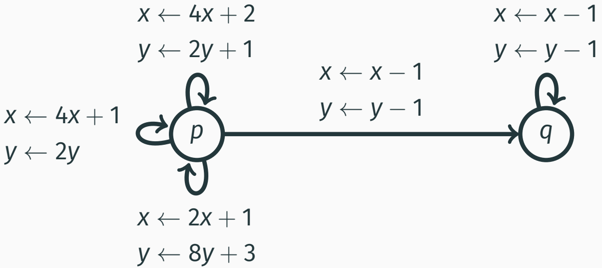

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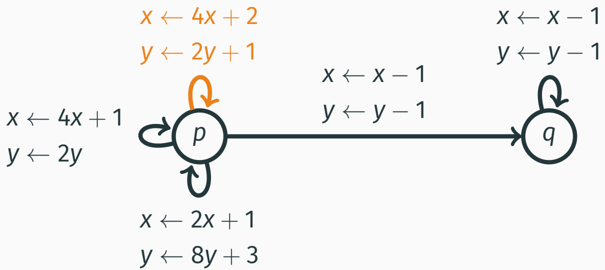

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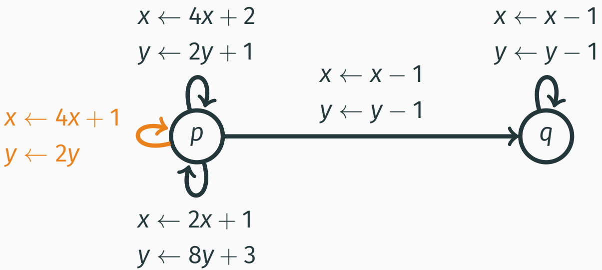

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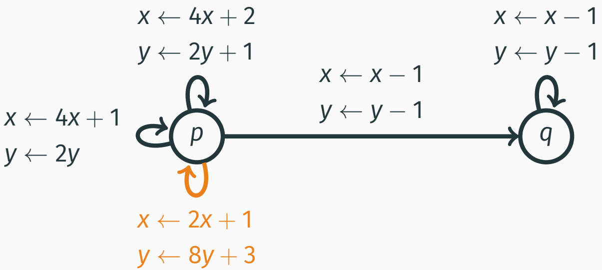

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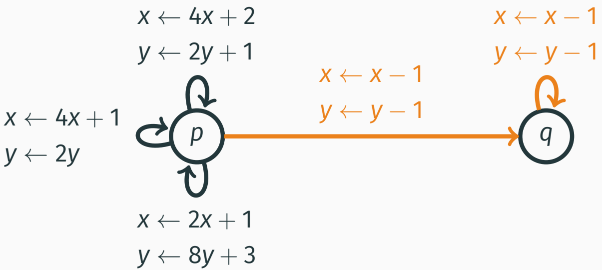

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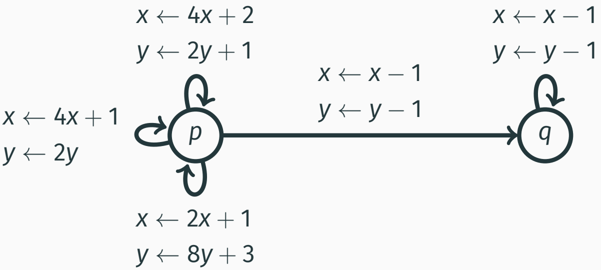

Has solution iff $p(1,1) \xrightarrow{*}_{\mathbb{Z}} q(1,1)$

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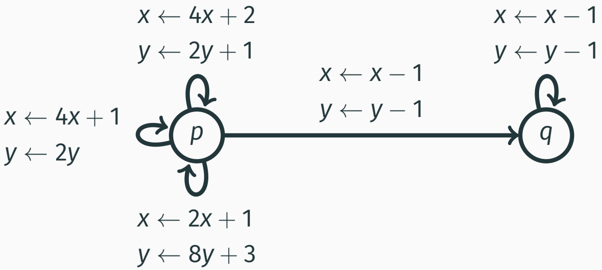
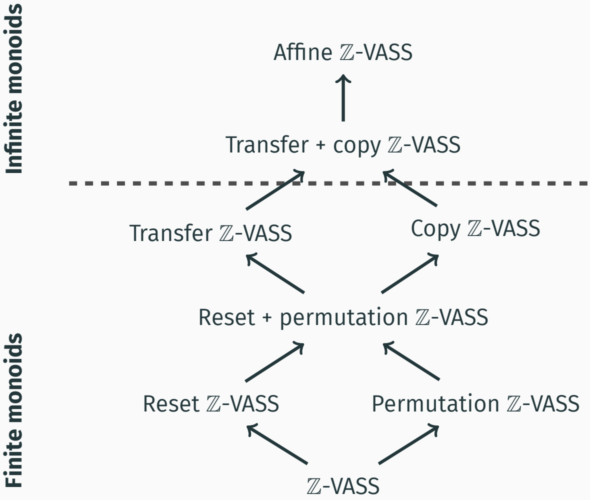


Doubling can be done with
a gadget of transfers and copies

Finite matrix monoids over $\mathbb{N}$



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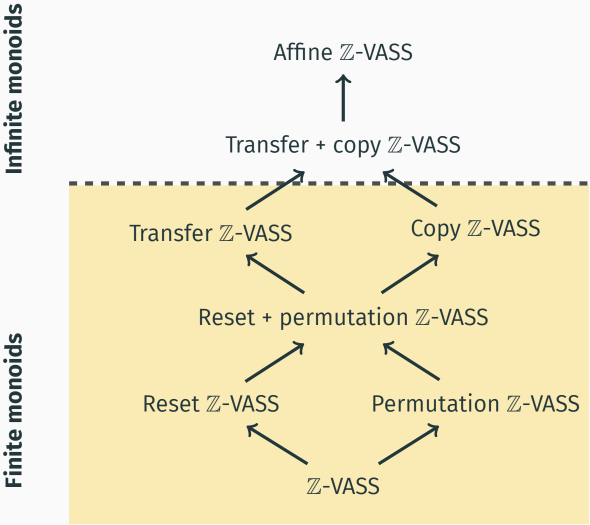

$\in$ **PSPACE**

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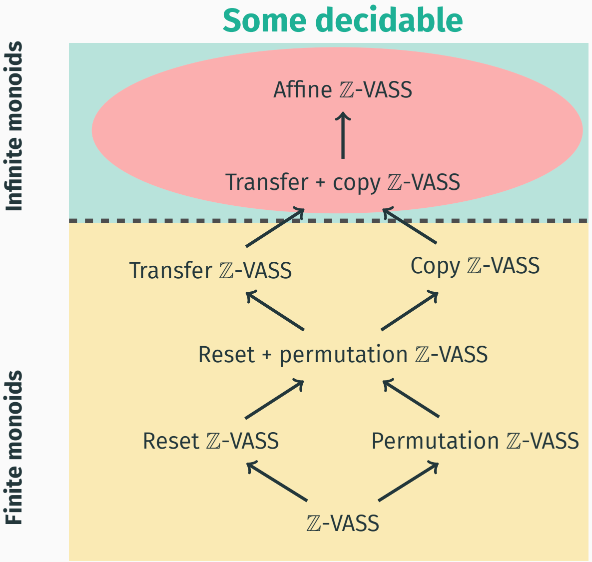

$\in$ **PSPACE**

Conclusion: summary

- Unified approach to reachability in affine $\mathbb{Z}$-VASS
- Possible to remove transformations when matrix monoid is finite
- Reachability relation of affine $\mathbb{Z}$-VASS is semilinear when monoid is finite
- Classification of complexity w.r.t. extensions

Conclusion: further work

- Complexity of reachability for permutation $\mathbb{Z}$-VASS?
- Size of matrix monoid for arbitrary affine $\mathbb{Z}$-VASS?
- Characterization of classes of infinite matrix monoids for which reachability is decidable?

Thank you!   Vielen Dank!